Fractional stochastic calculus in finance?

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0. Prolog

We all know that one should not use fractional Brownian motion in finance.
It is strictly forbidden.

But we also know that boys like to do forbidden things ...

Esko Valkeila $\sim$ 2010;
before starting a talk on fractional Brownian motion in finance ...

The background for Esko' statement was the following:

Recall the Samuelson model ~ 1970 for stock price $S(t)$ at time t:

$$dS(t) = bS(t)dt + \sigma S(t)dB(t); \quad S(0) > 0 \tag{1.1}$$

where $B(t) = B(t, \omega); t \geq 0, \omega \in \Omega$ is Brownian motion and the equation is interpreted as an Ito stochastic differential equation,. Here b and $\sigma \neq 0$ are constants.

It soon became clear that this model was not perfect, and people were looking for improvements. About 20 years later came the extension to the *fractional Brownian motion* model:

$$dS(t) = bS(t)dt + \sigma S(t)dB^H(t); \quad S(0) > 0 \tag{1.2}$$

where $B^H(t)$ is the fractional Brownian motion with Hurst coefficent $H \in (0, 1)$.
(Note that $B^{\frac{1}{2}}(t) = B(t)$.)

The problem now is that if $H \neq \frac{1}{2}$ this stochastic differential equation can be defined in several different ways, mathematically. The two most important of these are:

1. The forward (ω-wise) integral definition
2. The Wick-Ito-Skorohod definition

In the classical Brownain motion case ($H = \frac{1}{2}$) the two definitions give the same result.

The same problem follows us if we proceed to define the wealth V(t) generated by a self-financing portfolio $\varphi(t)$ by

$$dV(t) = \varphi(t)dS(t). \tag{1.3}$$

Then we have

1. The forward integral interpretation gives markets with *arbitrage*
2. The Wick-Ito-Skorohod interpretation leads to a mathematical finance market model without arbitrage, but with challenges when it comes to financial interpretation.

See

Hu & Ø. (2003): *"Fractional white noise calculus and applications to finance"* [6],

and the critique in

Björk & Hult (2005): *"A note on Wick products and the fractional Black-Scholes model"* [3].

It is on this background that many researchers (except some bold mathematicians like Esko Valkeila) considered fractional Brownian motion as off limit in finance.

Before proceeding, I want to point out that although researchers in finance and mathematicians have a lot to talk about, there is (in my experience) a cultural difference:

- Researchers in finance remain faithful to the established theories and do not easily get diverted into model experiments, while
- mathematicians like to play around with models, just to see what happens.

It is with this understanding that I, being a mathematician, present the following, which I hope will be received with goodwill (and forgiven, if necessary)...

My question is:

Can fractional stochastic calculus still be of interest in finance?

After the basic model introduced by Samuelson, there has been a lot of modifications, for example by replacing the constant volatility σ by a stochastic volatility, e.g. including a fractional Brownian motion component into σ.

In the following I will present a completely different fractional calculus approach to mathematical finance, namely through *fractal time differentiation*.

This approach is based on the fact that In many biological or transport systems the relation between the input flow $u(t)$ and the output reaction/growth rate $v(t)$ are of the form

$$v(t) = D^\alpha u(t) \tag{1.4}$$

where $D^\alpha = \frac{d^\alpha}{dt^\alpha}$ denotes the *(Abel-Caputo) fractional derivative operator* of order $\alpha \in (0, 2)$. ($\alpha = 1$ corresponds to the classical derivative: $D^1 = \frac{d}{dt}$)

If we adopt this point of view for the relation between the given price and its rate of change in the stock market (which can be regarded as a complicated mixture of a biological and a transport system), we get the equation

$$D^\alpha S(t) = bS(t) + \sigma S(t) \text{"}\textit{noise}\text{"} \tag{1.5}$$

where, as before, we represent "noise" by "white noise", i.e. the time-derivative of Brownian motion:

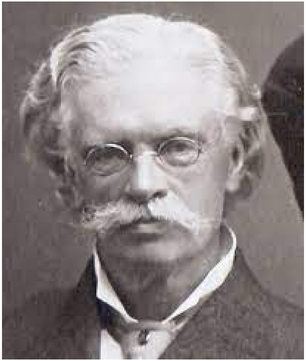
$$\text{"noise"} = W(t) := \frac{d}{dt} B(t). \tag{1.6}$$

Can we make rigorous mathematical sense of equations (1.5)-(1.6)?

1. Introduction

The *fractional derivative* of a function was first introduced by Niels Henrik Abel in 1823 [1], in connection with his solution of the tautochrone (isochrone) problem in mechanics.

The *Mittag-Leffler function* $E_\alpha(z)$ was introduced by Gösta Magnus Mittag-Leffler in 1903 [12]. He showed that this function has a connection to the fractional derivative introduced by Abel.



The fractional derivative turns out to be useful in many situations, e.g. in the study of waves, including ocean waves around an oil platform in the North Sea, and ultrasound in bodies. In particular, the fractional heat equation may be used to describe anomalous heat diffusion, and it is related to power law attenuation.
Many applications of fractional derivatives can be found in the book by S. Holm [7].

In this paper we study the following fractional stochastic differential equation

$$d^{\alpha}X(t) = bX(t)dt + \sigma X(t)dB(t); t \geq 0$$
$$X(0) = x > 0 \tag{2.1}$$

where $b, \sigma \neq 0$ are given constants in $\mathbb{R}$. We interpret the equation (2.1) as the following $(\mathcal{S})^*$-valued differential equation, which could be called *the fractional geometric Brownian motion equation*

$$\frac{d^{\alpha}}{dt^{\alpha}}X(t) = bX(t) + \sigma X(t) \diamond W(t)dt$$
$$X(0) = x \tag{2.2}$$

where

$$W(t) = W(t,\omega) = \frac{d}{dt}B(t) \tag{2.3}$$

is *white noise*, and

$$B(t) = B(t,\omega); t \geq 0, \omega \in \Omega$$

is Brownian motion with probability law $\mathbb{P}$.

- In the classical case, when $\alpha = 1$, we have $\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt}$ and equation (2.1) becomes the classical stochastic differential equation of Ito type, called the *geometric Brownian motion equation*:

$$\begin{aligned} dX(t) &= bX(t)dt + \sigma X(t)dB(t); t \geq 0 \\ X(0) &= x > 0 \end{aligned} \tag{2.4}$$

which, by Ito's formula, has the solution

$$X(t) = x \exp((b - \tfrac{1}{2}\sigma^2)t + \sigma B(t)); \quad t > 0 \tag{2.5}$$

(geometric Brownian motion)

Recall that this solution has the expected value

$$\mathbb{E}[X(t)] = xe^{bt}. \tag{2.6}$$

- When $\alpha > 1$ equation (2.1) models *superdiffusion or enhanced diffusion*, where the particles spread faster than in regular diffusion. This occurs for example in some biological systems.

- When $\alpha < 1$ the equation models *subdiffusion*, in which travel times of the particles are longer than in the standard case. Such situation may occur in transport systems.

We now ask the following:

Problem

Does the fractional geometric Brownian motion (2.2) give a good model in finance, for example for asset prices, for any value of $\alpha \neq 1$?

We consider the equation (2.2) in the sense of distributions, and we show that the unique $(\mathcal{S})^*$- valued solution $X(t)$ (where $(\mathcal{S})^*$ is the Hida space of stochastic distributions.) coincides with the solution of a stochastic linear Volterra equation. Then we discuss the properties of the solution and compare them for different values of α.

2. The space of tempered distributions

For the convenience of the reader we recall some of the basic properties of the Schwartz space $\mathcal{S}$ of rapidly decreasing smooth functions and its dual, the space $\mathcal{S}'$ of tempered distributions.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing smooth real functions f on $\mathbb{R}$ equipped with the family of seminorms:

$$\|f\|_{k,\beta} := \sup_{x\in\mathbb{R}} \left\{(1+|x|^k)|\partial^\beta f(x)|\right\} < \infty,$$

where $k = 0, 1, ...,$ $\beta = (\beta_1, ..., \beta_n)$ is a multi-index with $\beta_j = 0, 1, ...$ $(j = 1, ..., n)$ and

$$\partial^\beta f(x) := \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \cdots \partial x^{\beta_n}} f(x)$$

for $|\beta| = \beta_1 + ... + \beta_n$.

Then $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is a Fréchet space.
Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ be its dual, called the space of *tempered distributions*. Let $\mathcal{B}$ denote the family of all Borel subsets of $\mathcal{S}'(\mathbb{R})$ equipped with the weak* topology. If $\Phi \in \mathcal{S}'$ and $f \in \mathcal{S}$ we let

$$\Phi(f) \text{ or } \langle \Phi, f \rangle \tag{3.1}$$

denote the action of Φ on f.

Example

- (Evaluations) For $y \in \mathbb{R}$ define the function δ_y on $\mathcal{S}(\mathbb{R})$ by $\delta_y(\phi) = \phi(y)$. Then δ_y is a tempered distribution.
- (Derivatives) Consider the function D, defined for $\phi \in \mathcal{S}(\mathbb{R})$ by $D[\phi] = \phi'(y)$. Then D is a tempered distribution.
- (Distributional derivative)
  Let T be a tempered distribution, i.e. $T \in \mathcal{S}^{'}(\mathbb{R})$. We define the distributional derivative $T^{'}$ of T by

  $$T^{'}[\phi] = -T[\phi^{'}]; \quad \phi \in \mathcal{S}.$$

  Then $T^{'}$ is again a tempered distribution.

3. The Mittag-Leffler functions

Definition

(The two-parameter Mittag-Leffler function) The Mittag-Leffler function of two parameters $\alpha,\ \beta$ is denoted by $E_{\alpha,\beta}(z)$ and defined by:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{4.1}$$

where $z,\ \alpha,\ \beta \in \mathbb{C},\ Re(\alpha) > 0$ *and* $Re(\beta) > 0$, and Γ is the Gamma function.

Definition

(The one-parameter Mittag-Leffler function) The Mittag-Leffler function of one parameter α is denoted by $E_\alpha(z)$ and defined as;

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{4.2}$$

where $z,\ \alpha \in \mathbb{C},\ Re(\alpha) > 0$.

Remark

Note that $E_\alpha(z) = E_{\alpha,1}(z)$ *and that*

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{4.3}$$

4. The (Abel-)Caputo fractional derivative

In this section we present the definitions and some properties of the Caputo derivatives.

Definition

The (Abel-)Caputo fractional derivative of order $\alpha > 0$ of a function f is denoted by $D^\alpha f(x)$ or $\frac{d^\alpha}{dx^\alpha} f(x)$ and defined by

$$(5.1) \qquad D^\alpha f(x) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(u)du}{(x-u)^{\alpha+1-n}}; & n-1 < \alpha < n \\ \frac{d^n}{dx^n} f(x); & \alpha = n. \end{cases}$$

Here n is the smallest integer greater than or equal to α.
If f is not smooth these derivatives are interpreted in the sense of distributions.

Example

If $f(x) = x$ and $\alpha \in (0,1)$ then

$$D^{\alpha} f(x) = \frac{x^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}. \tag{5.2}$$

In particular, choosing $\alpha = \frac{1}{2}$ we get

$$D^{\frac{1}{2}} f(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{5.3}$$

5. Laplace transform of Caputo derivatives

Recall that the Laplace transform L is defined by

$$Lf(s) = \int_0^\infty e^{-st} f(t)dt =: \widetilde{f}(s) \tag{6.1}$$

for all f such that the integral converges.
Some of the properties of the Laplace transform that we will need are:

$$L[\frac{\partial^\alpha}{\partial t^\alpha} f(t)](s) = s^\alpha (Lf)(s) - s^{\alpha-1} f(0) \tag{6.2}$$

$$L[E_\alpha(bt^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha - b} \tag{6.3}$$

$$L[t^{\alpha-1} E_{\alpha,\alpha}(bt^\alpha)](s) = \frac{1}{s^\alpha - b} \tag{6.4}$$

Recall that the convolution $f * g$ of two functions $f, g : [0, \infty) \mapsto \mathbb{R}$ is defined by

$$(f * g)(t) = \int_0^t f(t-r)g(r)dr; \quad t \geq 0. \tag{6.5}$$

The convolution rule for Laplace transform states that

$$L\left(\int_0^t f(t-r)g(r)dr\right)(s) = Lf(s)Lg(s),$$

or

$$\int_0^t f(t-w)g(w)dw = L^{-1}\left(Lf(s)Lg(s)\right)(t). \tag{6.6}$$

6. White noise

Define $\Omega = \mathcal{S}'(\mathbb{R})$, equipped with the weak-star topology. This space will be the base of our basic probability space, which we explain in the following:

As events we will use the family $\mathcal{F} = \mathcal{B}(\mathcal{S}'(\mathbb{R}))$ of Borel subsets of $\mathcal{S}'(\mathbb{R})$, and our probability measure $\mathbb{P}$ is defined by the following result:

Theorem

(The Bochner–Minlos theorem)
There exists a unique probability measure $\mathbb{P}$ *on* $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ *with the following property:*

$$\mathbb{E}[e^{i\langle \cdot,\phi\rangle}] := \int_{\mathcal{S}'} e^{i\langle \omega,\phi\rangle}\mathbb{P}(d\omega) = e^{-\frac{1}{2}\|\phi\|^2}; \quad i = \sqrt{-1}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$*, where* $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R})}$ *and* $\langle \omega, \phi\rangle = \omega(\phi)$ *is the action of* $\omega \in \mathcal{S}'(\mathbb{R})$ *on* $\phi \in \mathcal{S}(\mathbb{R})$ *and* $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ *denotes the expectation with respect to* $\mathbb{P}$*.*

We call the triplet $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$ the *white noise probability space*, and $\mathbb{P}$ is called the *white noise probability measure*.

It is not difficult to prove that if $\phi \in L^2(\mathbb{R})$ and we choose $\phi_k \in \mathcal{S}(\mathbb{R})$ such that $\phi_k \to \phi$ in $L^2(\mathbb{R})$, then

$$\langle \omega, \phi \rangle := \lim_{k \to \infty} \langle \omega, \phi_k \rangle \quad \text{exists in} \quad L^2(\mathbb{P})$$

and is independent of the choice of $\{\phi_k\}$. In particular, if we define

$$\bar{B}(t) := \langle \omega, \chi_{[0,t]} \rangle; \quad t \geq 0,$$

then $\bar{B}(t, \omega)$ has an t-continuous version denoted by $B(t, \omega)$, which becomes a standard Brownian motion.

With this definition of Brownian motion it is natural to define the Wiener–Itô integral of $\phi \in L^2(\mathbb{R})$ by

$$\int_{\mathbb{R}} \phi(t) dB(t,\omega) := \langle \omega, \phi \rangle; \quad \omega \in \mathcal{S}'(\mathbb{R}).$$

We see that by using the Bochner–Minlos theorem we have obtained an easy construction of Brownian motion . Moreover, we get a representation of the space Ω as the Fréchet space $\mathcal{S}'(\mathbb{R}^d)$. This is an advantage in many situations, for example in the construction of the Hida-Malliavin derivative, which can be regarded as a stochastic gradient on Ω.

Since $B(t,\omega)$ is t-continuous a.s., we can for a.a. $\omega \in \Omega$ define its derivatives with respect to t in the sense of distributions. Thus we define the white noise $W(t) = W(t,\omega)$ by

$$W(t) = \frac{d}{dt}B(t) \text{ in } \mathcal{S}'. \tag{7.1}$$

The definition (7.1) can also be interpreted as an element of the Hida space $(\mathcal{S})^*$ of *stochastic distributions*, and in that setting it has been proved (see Lindstrøm, Ø. , Ubøe [11] and Benth [2]) that the Ito- integral with respect to $dB(t)$ can be expressed as

$$\int_0^T f(t,\omega)dB(t) = \int_0^T f(t,\omega) \diamond W(t)dt, \tag{7.2}$$

where $\diamond$ denotes *the Wick product*.

8. Relation to linear stochastic Volterra equations

We are now ready to prove the following result:

Theorem

The solution $X(t) \in (\mathcal{S})^*$ of the fractional geometric Brownian motion equation

$$d^\alpha X(t) = bX(t)dt + \sigma X(t)dB(t); t \geq 0$$

(8.1) $$X(0) = x > 0$$

coincides with the solution $X(t)$ of the following linear stochastic Volterra equation:

(8.2)

$$X(t) = xE_{\alpha,1}(bt^\alpha) + \sigma \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(b(t-u)^\alpha) X(u) dB(u).$$

Proof. a) Uniqueness.
We interpret the equation (8.1) as the $(\mathcal{S})^*$-valued differential equation

$$\frac{d^\alpha}{dt^\alpha} X(t) = bX(t) + \sigma X(t) \diamond W(t) dt$$
$$X(0) = x > 0. \tag{8.3}$$

First, suppose that $X(t)$ is a solution. We apply the Laplace transform L to both sides of (8.3) and obtain (see (6.2)):

$$s^{\alpha}\widetilde{X}(s) - s^{\alpha-1}X(0) = b\widetilde{X}(s) + \sigma\widetilde{X \diamond W}(s). \tag{8.4}$$

or,

$$(s^{\alpha} - b)\,\widetilde{X}(s) = s^{\alpha-1}x + \sigma\widetilde{X \diamond W}(s). \tag{8.5}$$

Hence

$$\widetilde{X}(s) = \frac{s^{\alpha-1}x}{s^{\alpha} - b} + \frac{\sigma}{s^{\alpha} - b}\widetilde{X \diamond W}(s) \tag{8.6}$$

Applying the inverse Laplace operator L^{-1} to this equation we get

$$X(t) = L^{-1}\Big(\frac{s^{\alpha-1}x}{s^\alpha - b}\Big)(t) + L^{-1}\Big(\frac{\sigma \widetilde{X \diamond W}(s)}{s^\alpha - b}\Big)(t)$$

$$= xE_{\alpha,1}(bt^\alpha) + L^{-1}\Big(\frac{\sigma \widetilde{X \diamond W}(s)}{s^\alpha - b}\Big)(t), \tag{8.7}$$

where we recall that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{8.8}$$

is the Mittag-Leffler function.

It remains to find $L^{-1}\left(\frac{\sigma\widetilde{X\diamond W}(s)}{s^\alpha - b}\right)(t)$
By (6.4) we have

$$L^{-1}\left(\frac{1}{s^\alpha - b}\right)(t) = t^{\alpha-1}E_{\alpha,\alpha}(bt^\alpha)$$

$$=: \Lambda(t). \tag{8.9}$$

In other words,

$$\frac{\sigma}{s^\alpha - b} = \sigma L\Lambda(s), \tag{8.10}$$

Combining with Lemma **??** we get

$$
\begin{aligned}
L^{-1}\left(\frac{\sigma}{s^{\alpha}-b}\widetilde{X\diamond W}(s)\right)(t) &= L^{-1}\left(L\left(\sigma\Lambda(s)\right)\widetilde{X\diamond W}(s)\right)(t) \\
&= \sigma\int_0^t \Lambda(t-u)X(u)\diamond W(u)du. \qquad (8.11)
\end{aligned}
$$

Substituting this into (8.7) leads to

$$
\begin{aligned}
X(t) &= xE_{\alpha,1}\left(bt^{\alpha}\right)+\sigma\int_0^t \Lambda(t-u)X(u)\diamond W(u)du \\
&= xE_{\alpha,1}\left(bt^{\alpha}\right)+\sigma\int_0^t (t-u)^{\alpha-1}E_{\alpha,\alpha}(b(t-u)^{\alpha})X(u)dB(u). \qquad (8.12)
\end{aligned}
$$

This proves uniqueness and also that the unique solution (if it exists) is given by the above formula (8.2).

b) Next, define $X(t)$ by (8.2). Then we can prove that $X(t)$ satisfies (8.1) by reversing the argument above. We skip the details. $\square$

Remark

- *Note that* (8.2) *is a linear stochastic Volterra equation in the unknown process* $X(t)$*. In particular, this shows that, unless* $\alpha = 1$*, this model for asset prices has memory. This is an interesting feature that is missing in the classical Samuelson/Black-Scholes model.*
- *Deterministic Volterra equations (i.e.* $\sigma = 0$*) with Mittag-Leffler functions as coefficients have been studied by Kilbas et al [9].*
- *From* (8.2) *we get directly that*

$$\mathbb{E}[X(t)] = xe^{bt}; \quad t \geq 0 \tag{8.13}$$

for all α*.*

- In the standard case when $\alpha = 1$ the solution of the Volterra equation (8.2) coincides with the classical geometric Brownian motion (2.5). Indeed, using the Ito formula it is easy to verify that

$$X(t) := x \exp((b - \tfrac{1}{2}\sigma^2)t + \sigma B(t)); \quad t \geq 0. \tag{8.14}$$

  solves (8.2) when $\alpha = 1$.
- It would be interesting to test this time-fractional Black-Scholes model in specific cases, for different values of α.

9. Properties of the solution

There are many unanswered questions about this solution $X(t)$. Some immediate partial results are:

Theorem

(Computation of the solution)

Suppose $\alpha > 1$.

Define

$F(t) = xE_{\alpha,1}(bt^{\alpha})$ *and* $K(v) = \sigma v^{\alpha-1} E_{\alpha,\alpha}(bv^{\alpha}); v \geq 0$.

Put $X^{(0)}(t) = F(t)$ *and define the processes* $X^{(n)}(t)$ *recursively as follows:*

(9.1)

$$X^{(n+1)}(t) = F(t) + \int_0^t K(t-s) X^{(n)}(s) dB(s); \quad n = 0, 1, 2, ...$$

Then $\{X^{(n)}\}_{n=1,2,...}$ *converges in* $\mathbb{L}^2([0,T] \times \Omega)$ *to a process* $X_{\alpha}(t)$*, which is the unique solution of* (8.2).

Theorem

(Positivity)

Suppose $\alpha < 1$.

Let $X_\alpha(t)$ *be the solution of* (8.2). *Then, for a.a.* ω, $X_\alpha(t)$ *converges uniformly in* $[0, T]$ *to*

$$X_1(t) = x \exp((b - \tfrac{1}{2}\sigma^2)t + \sigma B(t)) \tag{9.2}$$

as $\alpha \to 1^-$.

In particular, $X_\alpha(t) > 0$ *a.s., for all* $\alpha < 1$ *sufficiently close to 1.*

10. A fractional financial market

Coming back to fractional modelling in finance, in view of the general fractional relation (1.4) between input and output in biological/transport systems, we may heuristically interpret the fractional equation

$$D^{\alpha}X(t) = bX(t) + \sigma X(t) \diamond W(t) \tag{10.1}$$

or, in differential form,

$$d^{\alpha}X(t) = bX(t)dt + \sigma X(t)dB(t), \tag{10.2}$$

as the relation between the input

- $bX(t)dt + \sigma X(t)dB(t)$
  and the corresponding growth rate (output)
- $Y(t) = d^{\alpha}X(t)$.

Now let us apply this to a financial market, consisting of two assets

$$S(t) = (S_0(t), S_1(t)),$$

where S_0 denotes a risk-free asset, with unit price

(10.3) $$S_0(t) = 1 \quad \text{for all } t \geq 0,$$

and S_1 denotes a risky asset (e.g. stock) with the price dynamics

(10.4) $$d^{\alpha} S_1(t) = bS_1(t)dt + \sigma S_1(t)dB(t); \quad S_1(0) > 0.$$

Let

$$\varphi(t) = (\varphi_0(t), \varphi_1(t))$$

be a portfolio in this market, giving the number of units held at time t in the assets S_0 and S_1, respectively.

The value at time t of this portfolio is defined by

$$V(t) = \varphi_0(t)S_0(t) + \varphi_1(t)S_1(t) = \varphi(t)S(t) \quad \text{(inner product)}.$$

In the classical case, with $\alpha = 1$, we say that φ is self-financing if

$$dV(t) = \varphi(t)dS(t)$$

(Intuitively, the infinitesimal change in the value V comes from the infinitesimal change in S only.)

Similarly, in the fractional market, we say that the portfolio is self-financing if we have a corresponding relation between the growth rate $d^\alpha V(t)$ and the growth rate $d^\alpha S(t)$, i.e.

$$\begin{aligned} d^\alpha V(t) &= \varphi(t)d^\alpha S(t) \\ &= \varphi_0(t)d^\alpha S_0(t) + \varphi_1(t)d^\alpha S_1(t) \\ &= \varphi_1(t)d^\alpha S_1(t) \\ &= \varphi_1(t)[bS_1(t)dt + \sigma S_1(t)dB(t)] \end{aligned}$$

This gives, after some calculations, the following formula for the value process:

$$V(t) = V(0) + \Gamma(\alpha) \int_0^t \varphi_1(s) S_1(s)(t-s)^{\alpha-1}[b ds + \sigma dB(s)].$$

If we, as is customary, define $\pi(t)$ to be the fraction of the value invested in the risky asset, then we have

$$\pi(t)V(t) = \varphi_1(t) S_1(t)$$

and the equation above gets the form

(10.5)

$$V(t) = V(0) + \Gamma(\alpha) \int_0^t \pi(s) V(s)(t-s)^{\alpha-1}[b ds + \sigma dB(s)].,$$

which is, for given portfolio π, a Volterra equation in V.

The above market leads to many interesting questions that I would have liked to discuss with Tomas. For example:

Question 1: Does this market have an arbitrage?

Question 2: Is the market complete?

We see that last question is equivalent to the question of the existence of a solution of a backward stochastic fractional Volterra integral equation (BSVIE), namely the above equation (10.5) combined with a given required terminal value $V(T)$.

Those were the days, Tomas!

On our way to Guanajuato, shortly before the Montezuma revenge hit us!

Photo: Eva Øksendal Mexico City 18 March 1998

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