Immunization with consistent term structure dynamics

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Tomas

1995, Fixed-Income Conference, Hindsgavl, Denmark

- Heath-Jarrow-Morton model and curve shape
- System, control

1999, Interest Rate Dynamics and Consistent Forward Rate Curves, *Mathematical Finance*

- Freddy, Damir, Rolf, Josef, Jørgen Aase Nielsen

2000s, PhD courses in Aarhus, defences

- Relay races, families, ...
- Knowledge sharing, showing work

Immunization

Cross-sectional strategies, duration matching

Hedging performance depends on returns, dynamics

Multiple factors, generalized duration matching

Many parameters, estimation uncertainty

Parsimony, level-slope-curvature

Inconsistency (Björk and Christensen, 1999; Filipovic, 1999)

Curve shape will change under dynamic model

Strategy should rely on shape relevant for future curves, returns

Consistency

Curve shape loadings span DTSM drifts and volatilities

Distinct from no-arbitrage condition

Joint hypothesis

Loadings span volatilities, convexity, slope adjustments (spread, local slope)

Convexity quadratic in volatilities, hence loadings

More loadings for spanning

Convexity already spanned, slope adjustments must be spanned

Hedging framework

Yield factor model

Target and instruments

Optimal portfolio minimizing conditional hedging error variance

Parsimonious curve shape

Three approaches to consistency

1. More loadings for consistency
2. Excess returns (current curve and slope adjustments from data, reduced number of factors)
3. Filtering (imposing curve shape and consistent dynamics)

Results

Weekly Fed yields, eight maturities, 1983-2019

2, 5, 10-year coupon bond portfolio, $(-1, 3, -1)$

Classical three-factor analysis little gain over duration matching

Nelson and Siegel (1987) (NS) worse

Inconsistent

Consistency

Approach 1: Augmented NS (ANS) improves over NS and unrestricted factor model

Approach 2: Unrestricted return model worse, ANS-extended Vasicek (1977) DTSM better

Approach 3: Filtering worse

- Current curve not (yet) ANS, or
- DTSM too restrictive (one stochastic factor - slope)

No-arbitrage condition rejected

Stochastic level, slope, curvature

Volatilities proportional to NS loadings (Vasicek only second)

Seven loadings in consistent curve shape

Three stochastic factors, four deterministic

AFNS (Christensen, Diebold and Rudebusch, 2011) for deterministic factors at long-run levels

SLSC strongest performance

Approach 3 as strong as Approach 2

No-arbitrage condition not rejected

MCS is SLSC

AFNS worse

Interpretation

Alternatives to duration matching

- Generalized duration matching not sufficient
- Neither is basic parsimony
- Consistency does the job

Yield curve shape consistent with driving stochastic process

Not standard affine (AFNS)

Intercept replaced by time-varying mixture of loadings

Yield factor model

ZCB yields, terms to maturity $\tau_1, \ldots, \tau_m$

$$y_t = \mu + B f_t + \varepsilon_t\,, \quad \text{var}(\varepsilon_t) = \Psi$$

B is $m \times k$

Returns

$$r_{t+1,\tau_i} \approx \log \frac{p_{t+1,\tau_i}}{p_{t,\tau_i}} = -\tau_i \Delta y_{t+1,\tau_i} = -\tau_i (b_i \Delta f_{t+1} + \Delta \varepsilon_{t+1,i})$$

b_i is ith row of B

Generalized durations $\tau_i b_i$

Optimal portfolio weights

Theorem 1
The immunization portfolio w_* that minimizes total hedging error variance subject to generalized duration and value matching

$$\min_{w} \mathrm{var}_t \left[r^*_{t+1} - w' r_{t+1} \right] \quad s.t. \quad w' \mathcal{T} B = (\tau b)_* \quad and \quad w' \iota = 1$$

is given by

$$w_* = \widetilde{w} + \left(1 - \widetilde{w}' \iota\right) \frac{\Lambda \iota}{\iota' \Lambda \iota}, \quad \widetilde{w} = \mathcal{T}^{-1} \Psi^{-1} B \left(B' \Psi^{-1} B\right)^{-1} (\tau b)'_* \qquad (1)$$

where

$$\begin{aligned} \Lambda &= \mathcal{T}^{-1} \Psi^{-1} \left(\Psi - B \left(B' \Psi^{-1} B\right)^{-1} B'\right) \Psi^{-1} \mathcal{T}^{-1}, \\ \mathcal{T} &= \mathrm{diag}(\tau_1, \ldots, \tau_m) \end{aligned}$$

Data and benchmark

- Sample period is 1983 - 2019
- Target
  - CRSP Monthly Treasury files
  - Portfolio of (2,5,10)-year coupon bonds, $(-1, 3, -1)$
- Instruments and model estimation
  - Fed's database of constant maturity zero-coupon yields
  - Terms to maturity 3, 6, 12, 24, 36, 60, 84, 120 months

Data

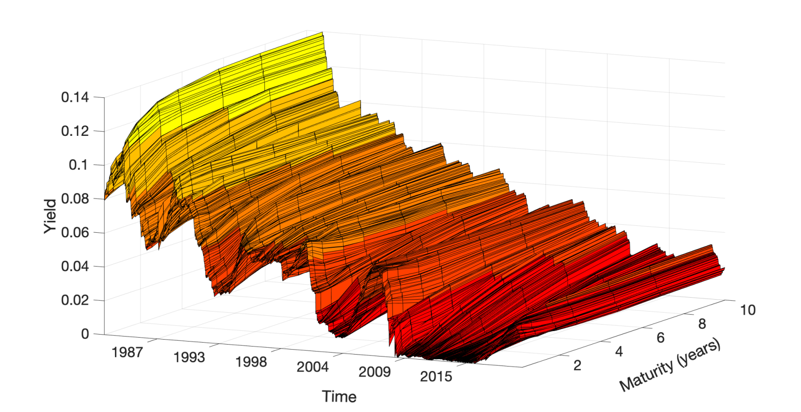

Figure 1: Three-dimensional view of the weekly yield data from January, 1983, through December, 2019, at maturities 0.25, 0.5, 1, 2, 3, 5, 7, and 10 years.

Immunization performance

Classical factor analysis

$$y_t = \mu + Bf_t + \varepsilon_t$$

	Model	Bias	Std. dev.	**RMSE**	MAE
1	Target movement	53.84	143.95	153.52	119.94
2	Duration matching	8.03	55.67	56.18	40.73
3	Unrestricted 3-factor Full period	4.32	47.40	47.53	34.45
4	Unrestricted 3-factor Rolling 4-year	4.03	46.68	46.79 (S,-)	34.13 (S,-)

Table 1: Immunization performance. Results are in basis points per month.

Nelson-Siegel for parsimony

B restricted

$$y_t = \mu + B f_t + \varepsilon_t,$$

Level, slope, curvature structure

$$B_{1:3}(\tau) = \left(1, \quad \frac{1 - e^{-a\tau}}{a\tau}, \quad \frac{1 - e^{-a\tau}}{a\tau} - e^{-a\tau}\right)$$

Immunization performance

	Model	Bias	Std. dev.	**RMSE**	MAE
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4	Unrestricted 3-factor Rolling 4-year	4.03	46.68	46.79 (S,-)	34.13 (S,-)
5	Nelson-Siegel Full period	3.91	48.19	48.29	34.86
6	Nelson-Siegel Rolling 4-year	4.34	49.74	49.87 (S,-)	36.11 (S,-)

Table 2: Immunization performance. Results are in basis points per month.

Heath, Jarrow and Morton (1992)

$$
\begin{aligned}
dy(t,\tau) &= \alpha(t,\tau)dt + \sigma(t,\tau)' dW_t \\
\alpha(t,\tau) &= \frac{1}{\tau}(y(t,\tau) - y(t,0)) + \frac{\partial y}{\partial \tau}(t,\tau) + \frac{\tau}{2}\sigma(t,\tau)'\sigma(t,\tau) + \sigma(t,\tau)'\lambda_t
\end{aligned}
$$

DTSM (α,σ), or (λ,σ), if arbitrage-free

Consistency

- Consider a class of potential yield curves $Y(\tau, x)$, parametrized by $x \in \mathcal{X} \subseteq \mathbb{R}^k$, i.e., the class is $\mathcal{Y} = \{Y(\cdot, x) | x \in \mathcal{X}\}$
- Let $T_{\mathcal{Y}} = \inf_t \{t : \exists x \in \mathcal{X} \text{ s.t. } y(t, \cdot) = Y(\cdot, x)\}$ be the first hitting time for $\mathcal{Y}$

Consistency

Definition

(a) **Dynamic consistency** between a DTSM (α, σ) and a given class $\mathcal{Y}$ of potential yield curves means that if the yield curve dynamics are governed by the DTSM, then $y(t, \cdot) \in \mathcal{Y}$, for $t \geq T_{\mathcal{Y}}$

(b) **Strong dynamic consistency** between a DTSM (α, σ) and $\mathcal{Y}$ means that if the yield curve dynamics are governed by the DTSM, then $y(t, \tau) = Y(\tau, x_t)$, for $t \geq T_{\mathcal{Y}}$, with

$$\mathrm{d}x_t = \phi(t)\mathrm{d}t + \psi(t)'\mathrm{d}W_t\,,$$

for suitable ϕ, ψ

Consistency

If the curve shape $\mathcal{Y}$ is linear in x, i.e., $Y(\tau,x) = B(\tau)x$, with $B(\tau)$ $1 \times k$, then we have a factor model

Assume $\sigma(t,\tau) = \sigma(\tau,x)$, hence $\phi(t) = \phi(x)$, $\psi(t) = \psi(x)$

Consistency between the arbitrage-free DTSM (λ,σ) and $\mathcal{Y}$ is equivalent to the existence of suitable ϕ, ψ satisfying the conditions

$$
\begin{aligned}
\frac{1}{\tau}\left[B(\tau) - B(0)\right]x \quad &+ \quad \frac{\partial B}{\partial \tau}(\tau)x + \frac{\tau}{2}B(\tau)\psi(x)'\psi(x)B(\tau)' \\
&= \quad B(\tau)(\phi(x) - \psi(x)'\lambda(x)), \\
\sigma(\tau,x)' \quad &= \quad B(\tau)\psi(x)'
\end{aligned}
$$

for all (τ,x)

Loadings must span convexity as well as spread and local slope

Consistency

NS is inconsistent with any non-degenerate arbitrage-free DTSM

The augmented NS (ANS) curve shape given by loading functions

$$B(\tau) = \begin{pmatrix} 1 & \dfrac{1-e^{-a\tau}}{a\tau} & \dfrac{1-e^{-a\tau}}{a\tau} - e^{-a\tau} & \dfrac{1-e^{-2a\tau}}{2a\tau} \end{pmatrix}$$

is consistent with the arbitrage-free DTSM with volatility function

$$\sigma(t,x)' = \sigma_2 \left(\frac{1-e^{-a\tau}}{a\tau} \right)$$

Approach 1: Impose consistent curve shape on B in factor analysis

Immunization performance

	Model	Bias	Std. dev.	**RMSE**	MAE
1	Target movement	53.84	143.95	153.52	119.94
2	Duration matching	8.03	55.67	56.18	40.73
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5	Nelson-Siegel Full period	3.91	48.19	48.29	34.86
6	Nelson-Siegel Rolling 4-year	4.34	49.74	49.87 (S,-)	36.11 (S,-)
7	Unrestricted 4-factor Full period	4.61	28.80	29.13	22.56
8	Unrestricted 4-factor Rolling 4-year	3.21	36.39	36.48 (S,-)	24.38 (S,-)
9	ANS Approach 1 Full period	3.69	29.98	30.17	22.67
10	ANS Approach 1 Rolling 4-year	2.41	26.32	26.39 (S,-)	16.89 (S,-)

Table 3: Immunization performance. Results are in basis points per month.

Consistency

Proposition

The SLSC curve shape given by loading functions

$$B(\tau) = \left(\frac{1-e^{-b\tau}}{b\tau}, \quad \frac{1-e^{-a\tau}}{a\tau}, \quad \frac{1-e^{-a\tau}}{a\tau} - e^{-a\tau}, \quad \frac{1-e^{-2b\tau}}{\tau}, \quad \frac{1-e^{-2a\tau}}{\tau}, \quad e^{-2a\tau}, \quad \tau e^{-2a\tau} \right)$$

is consistent with the DTSM with volatility function

$$\sigma(\tau, x)' = \left(\sigma_1 \left(\frac{1-e^{-b\tau}}{b\tau} \right), \quad \sigma_2 \left(\frac{1-e^{-a\tau}}{a\tau} \right), \quad \sigma_3 \left(\frac{1-e^{-a\tau}}{a\tau} - e^{-a\tau} \right) \right)$$

for $b > 0$ small (to avoid non-stationary factor and linearly increasing loading for spanning convexity)

Consistency

$$\Phi = \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -a & 0 & -1 & 0 & 0 \\ 0 & 0 & a & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2a & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 2a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2b \end{pmatrix}$$

and

$$\theta = \begin{pmatrix} \frac{1}{b^2}\psi_{11}^2 + \frac{1}{b}\lambda_1\psi_{11} \\ \frac{1}{4a^2}(4\omega_{22} + 7\omega_{33} + 10\omega_{23}) + \frac{1}{a}\left(\lambda_2\psi_{22} + (\lambda_2 + \lambda_3)\psi_{23} + \lambda_3\psi_{33}\right) \\ \frac{1}{4a^2}(\omega_{33} + 2\omega_{23}) + \frac{1}{a}\left(\lambda_2\psi_{23} + \lambda_3\psi_{33}\right) \\ -\frac{1}{4a^2}(2\omega_{22} + 5\omega_{33} + 6\omega_{23}) \\ \frac{1}{2a}\omega_{33} \\ -\frac{1}{4a}(3\omega_{33} + 2\omega_{23}) \\ -\frac{1}{2b^2}\psi_{11}^2 \end{pmatrix}$$

Approach 2

$$dy(t,\tau) = \alpha(t,\tau)dt + \sigma(t,\tau)' dW_t$$

$$\alpha(t,\tau) = \frac{1}{\tau}(y(t,\tau) - y(t,0)) + \frac{\partial y}{\partial \tau}(t,\tau) + \frac{\tau}{2}\sigma(t,\tau)'\sigma(t,\tau) + \sigma(t,\tau)'\lambda_t$$

Slope-adjusted yield changes

$$\tilde{y}(t+1,\tau) = \Delta y(t+1,\tau) - \frac{1}{\tau}(y(t,\tau) - y(t,0)) - \frac{\partial y}{\partial \tau}(t,\tau)$$

where excess returns $r(t+1,\tau) - y(t,0) = -\tau\tilde{y}(t+1,\tau)$

For ψ, λ constant, only slope adjustments depend on x in α, so for $\phi(x) = \Phi(\theta - x)$, writing RHS as $B(\tau)\phi(x)$ allows

$$\tilde{y}(t+1,\tau) = B(\tau)\left(\Phi\theta + \psi' w_{t+1}\right) + \nu_\tau + \varepsilon_{t+1,\tau}$$

Curve and slope adjustments at t from data

Test $\nu_\tau = 0$ (no-arbitrage condition)

Approach 3

$$y(t,\tau) = B(\tau)x(t) + \varepsilon(t,\tau)$$
$$dx(t) = \phi(x(t))dt + \psi(x(t))'dt$$

Curve shape imposed throughout

B consistent with suitable DTSM (λ, σ) through coefficients ϕ, ψ

Explicit restrictions on $\phi = \Phi(\theta - x)$, ψ for ANS-extended Vasicek and SLSC

Filtering along consistent curve family

Factors (state variables) dynamic, unlike in classical factor analysis

Freezing (four) deterministic factors at long-run levels generates $A(\tau) = B_{4:7}(\tau)\theta_{4:7}$, performance deteriorates

Immunization performance

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	⋮				
9	ANS Approach 1 Full period	3.69	29.98	30.17	22.67
10	ANS Approach 1 Rolling 4-year	2.41	26.32	26.39 (S,-)	16.89 (S,-)
	⋮				
13	ANS Approach 2 Full period	1.64	15.89	15.96	12.06
14	ANS Approach 2 Rolling 4-year	0.56	22.41	22.39 (S,-)	15.28 (S,-)
	⋮				
23	SLSC Approach 3 Full period	0.72	9.07	9.09	7.11
24	SLSC Approach 3 Rolling 4-year	0.86	8.32	8.36 (S,MCS)	6.40 (S,-)

Table 4: Immunization performance. Results are in basis points per month.

Approach 2 better than 3 for ANS, not for SLSC, arbitrage for ANS

Estimated SLSC model

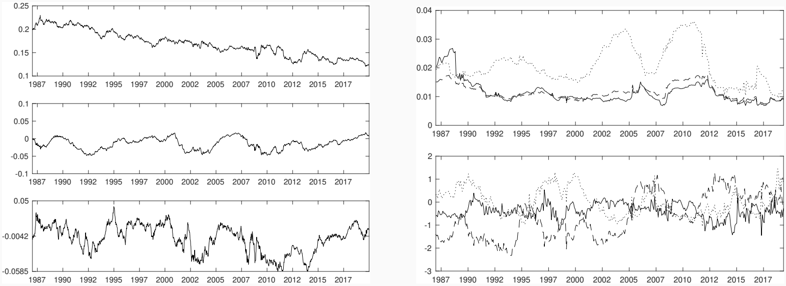

Figure 2: Time series evolution of fitted factors (left), estimated volatility parameters (upper right), and estimated market prices of risk (lower left) from filtering along the dynamically consistent curve family in the SLSC model.

Fitted yield curves

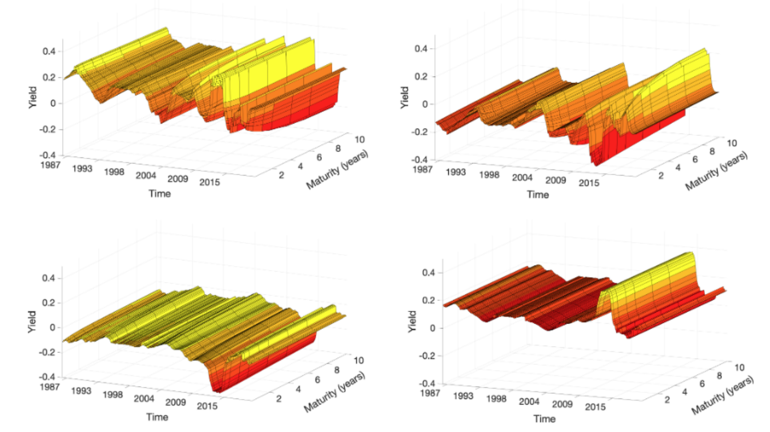

Figure 3: Stochastic (left) and deterministic (right) components of yield curve in unrestricted (top) and restricted (bottom) SLSC model.

Conclusion

Alternatives to duration matching

- Generalized duration matching not sufficient
- Neither is basic parsimony
- Consistency does the job

Stochastic level, slope, curvature model best

Yield curve shape consistent with driving stochastic process

Not standard affine (AFNS)

Intercept replaced by time-varying mixture of loadings

Performance deteriorates under alternative RMSE strategy

Parsimony again, weights depend on factor dynamics

We show the economic value of consistency - curve shape should be relevant in future