Term Structure Modeling under Entropy Maximization

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Conference in Memory of Tomas Björk 10/10/2022

Aim

Need for realistic long-term interest rate term structure model.

Maximize entropy for flexible market dynamics!

Derive interest rate dynamics!

Kelly (1956) maximized entropy of portfolio for given market dynamics.

Reference Market

Benchmark approach, Pl.& Heath (2006):

$(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$

n driving independent **Brownian motions** $W_t^1, ..., W_t^n$

m strictly positive **primary security accounts** $S_t^1, S_t^2, ..., S_t^m$

$$dS_t^j = S_t^j \left(\mu_t^j dt + \sum_{k=1}^{n} \sigma_t^{j,k} dW_t^k \right),$$

Growth Optimal Portfolio (GP)

No-arbitrage condition: GP $V_t^{\delta^*}$ exists.

Equivalent to NUPBR condition by Karatzas & Kardaras (2007).

NFLVR condition by Delbaen & Schachermayer (1998) is more restrictive.

Flilipovic & Pl.(2007): $\rightarrow$ **GP $V_t^{\delta^*}$ exists if and only if** π_t^* and λ_t^* solve

$$\begin{pmatrix} \sigma_t \sigma_t^\top & \underline{1} \\ \underline{1}^\top & 0 \end{pmatrix} \begin{pmatrix} \pi_t^* \\ \lambda_t^* \end{pmatrix} = \begin{pmatrix} \mu_t \\ 1 \end{pmatrix}$$

$\rightarrow$

$$\frac{dV_t^{\delta^*}}{V_t^{\delta^*}} = \lambda_t^* dt + \theta_t^\top (\theta_t dt + dW_t)$$

risk adjusted return

$$\lambda_t^* = \mu_t^\top \pi_t^* - \pi_t^{*\top} \sigma_t \sigma_t^\top \pi_t^*$$

internal market price of risk vector

$$\theta_t = \sigma_t^\top \pi_t^* .$$

Dynamics of internal market prices of risk θ_t^k ?

Basis securities $B_t^1, ..., B_t^n$:

$$\frac{d\frac{B_t^j}{V_t^{\delta *}}}{\frac{B_t^j}{V_t^{\delta *}}} = -\theta_t^j dW_t^j \tag{1}$$

j-th **squared internal market price of risk**

$$(\theta_t^j)^2 = a_t^j \phi^j(Y_{\tau_t^j}^j)$$

(stationary, Markovian)

activity $a_t^j > 0$

activity time

$$\tau_t^j = \tau_0^j + \int_0^t a_u^j du,$$

j-th **normalized index**

$$Y_{\tau_t^j}^j = \frac{\frac{V_t^{\delta^*}}{B_t^j}}{4e^{\tau_t^j}}$$

q^j **stationary density** of $Y_\cdot^j$,

f_t^j: **putative pricing density** for internal market price of risk θ_t^j

Radon-Nikodym derivative:

$$\Lambda_{\theta^j}(t) = \frac{\frac{B_t^j}{V_t^{\delta_*}}}{\frac{B_0^j}{V_0^{\delta_*}}},$$

Entropy Maximization

$q = \prod_{j=1}^{n} q^j$

$f_t = \prod_{j=1}^{n} f_t^j$

Joint entropy :

$$\mathcal{H}_n(q, f_t) = -\frac{1}{n} \int q(y) \ln(f_t(y)) dy = \mathcal{H}_n(q) - \mathcal{D}_n(q, f_t) \to max$$

$\mathcal{H}_n(q) = \mathcal{H}_n(q, q)$ *entropy* of q,

$\mathcal{D}_n(q, f_t)$ *Kullback-Leibler divergence* of q from f_t,

Determining joint entropy maximizing market dynamics.

Normalized Index

$$\mathbf{E}^q(\ln(Y^j_\cdot)) = \zeta_j$$

$$\mathbf{E}^q(Y^j_\cdot) = \bar{Y}_j$$

Lagrangian ... $\rightarrow$

j-th **normalized index:**

$$dY^j_{\tau^j_t} = (2e^{\gamma_E - 1} - Y^j_{\tau^j_t})d\tau^j_t + \sqrt{Y^j_{\tau^j_t} 2e^{\gamma_E - 1}}\, d\bar{W}^j_{\tau^j_t},$$

square root process of dimension 4,

$\gamma_E \approx 0.5772$ Euler-Mascheroni constant

Noether's Theorem, Noether (1918):
symmetries $\longleftrightarrow$ conserved quantities

j-th squared market price of risk

$$(\theta_t^j)^2 = \frac{a_t^j 2 e^{\gamma_E - 1}}{Y^j_{\tau_t^j}}$$

3/2 volatility model Pl.(1997), Heston(1997),

minimal market model (MMM)
Pl.(1997, 2001)

Market Activity

Minimizing Kullback-Leibler divergence $\rightarrow$

Activities converge toward each other:

$$\sqrt{a_t^j} \rightarrow \sqrt{a_t} = \frac{1}{n} \sum_{k=1}^{n} \sqrt{a_t^k}$$

for $j \in \{1, ..., n\}$

Market prices of risk on average equal!

Stocks

j-th stock

$$\frac{dS_t^j}{S_t^j} = \mu_t^j dt + \sigma_t^j dW_t^j + \sigma_t^* dW_t^*$$

GP of goods, services, hours of labour etc., consumed and/or produced by j-th company

$$\frac{dS_t^j}{S_t^j} = \lambda_t^j dt + \sigma_t^j (\sigma_t^j dt + dW_t^j) + \sigma_t^* (\sigma_t^* dt + dW_t^*)$$

internal market price of risk:

$$\sigma_t^j = \frac{\mu_t^j - (\sigma_t^*)^2 - \lambda_t^j}{\sigma_t^j} \approx \sqrt{a_t}$$

Adding savings account without change in average of market prices of risk:

$$\frac{dS_t^j}{S_t^j} = r_t dt + \sigma_t^j(\theta_t^j dt + dW_t^j) + \sigma_t^*(\sigma_t^* dt + dW_t^{j*})$$

$$\theta_t^j = \frac{\mu_t^j - (\sigma_t^*)^2 - r_t}{\sigma_t^j} \approx \sqrt{a_t}$$

$\rightarrow$ minimizing

$$\sum_{j=1}^{n}(\theta_t^j - \sqrt{a_t})^2$$

$$= \sum_{j=1}^{n}\left(\frac{\mu_t^j - (\sigma_t^*)^2 - r_t}{\sigma_t^j} - \sqrt{a_t}\right)^2$$

$\rightarrow$ **real interest rate:**

$$r_t = \frac{\sum_{j=1}^{n} \frac{\mu_t^j - (\sigma_t^*)^2 - \sqrt{a_t}\sigma_t^j}{(\sigma_t^j)^2}}{\sum_{j=1}^{n} \frac{1}{(\sigma_t^j)^2}}$$

Real Interest Rate

$$r_t = \tilde{\mu}_t - \sqrt{a_t}\tilde{\sigma}_t - (\sigma_t^*)^2 \approx \tilde{\mu}_t - a_t - a_t 2e^{\gamma_E - 1}\frac{1}{Y^*_{\tau_t^*}}$$

3/2 interest rate type model because of

$$d\frac{1}{Y^*_{\tau_t^*}} = \frac{1}{Y^*_{\tau_t^*}} d\tau_t^* + (\frac{1}{Y^*_{\tau_t^*}})^{\frac{3}{2}} \sqrt{2e^{\gamma_E - 1}} dW^*_{\tau_t^*}$$

Carr & Sun (2007), Goard & Mazur (2013)

real world pricing $\rightarrow$ **real world interest rate term structure**

1. **Law of No Arbitrage**:
GP exists.

2. **Law of Action and Reaction**:
For each volatility exists opposite volatility.

3. **Law of Inertia**:
$Y^j_{\cdot}$ square root process of dimension four.

4. **Law of Activity**:
Averages of market prices of risk converge toward each other.

5. **Law of Energy**:
Kinetic and potential energy.

6. **Law of the Minimal Price**:
Real World Pricing Formula.

6. **Law of Zero Activity**: