A Hereditary Hsu–Robbins–Erdös Law of Large Numbers

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Strong Law of Large Numbers Kolmogorov (1930)

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider independent, integrable, real-valued functions $f, f_1, f_2, \ldots$ with the same distribution. Then the CESÀRO means

$$\frac{1}{N}\sum_{n=1}^{N}, \quad N \in \mathbb{N}$$

converge $\mathbb{P}$-a.e., and with respect to the norm of $\mathbb{L}^1$, as $N \to \infty$, to the ensemble average

$$\mathbb{E}(f) = \int_{\Omega} f(\omega)\mathbb{P}(d\omega).$$

In 1947, P.L. Hsu and H.E. Robbins showed that, in the same setting but now under the stronger condition

$$\mathbb{E}(f^2) = \int_\Omega f^2(\omega)\mu(d\omega) < \infty$$

we have the stronger convergence, called complete convergence,

$$\sum_{N\in\mathbb{N}} \mathbb{P}(|\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f)| > \epsilon) < \infty$$

for every $\epsilon > 0$.

Then in 1949/50, P. Erdös showed that the square-intgrability $\mathbb{E}(f^2) < \infty$ is not only sufficient for this strengthening of the SLLN, but also necessary.

Here is a useful way to look at these results. We look at the sojourn times

$$T_\epsilon = \#\{N \in \mathbb{N} : |\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f)| > \epsilon\}$$

spent by the sequence of CESÀRO averages outside the interval

$$[\mathbb{E}(f) - \epsilon, \mathbb{E}(f) + \epsilon]$$

for $\epsilon > 0$. Then the SLLN amounts to

$$\mathbb{E}(|f|) < \infty \Rightarrow \mathbb{P}(T_\epsilon < \infty) = 1, \quad \forall \epsilon > 0.$$

The Hsu-Robbins result amounts to

$$\mathbb{E}(f^2) < \infty \Rightarrow \mathbb{E}(T_\epsilon) < \infty, \quad \forall \epsilon > 0.$$

And the ERDÖS result to the statement that the above implication goes also the other way.

A quantitative version due to C. Heyde

Under the Condition $\mathbb{E}(f^2) < \infty$ and with

$$\sigma^2 \triangleq \mathrm{Var}(f) = \mathbb{E}(f - \mathbb{E}(f))^2$$

we have also

$$\lim_{\epsilon \downarrow 0} (\epsilon^2 \ \mathbb{E}(T_\epsilon)) = \sigma^2$$

from HEYDE (1974).

Beyond the i.i.d.case

In 1967, J. Komlos proved a truly astonishing result.
For ANY sequence $f_1, f_2, \dots$ of integrable functions which are bounded in $\mathbb{L}^1$, i.e.,

$$\sup_{n \in \mathbb{N}} \mathbb{E}(|f_n|) < \infty,$$

there is an integrable f_* and a subsequence $f_{k_1}, f_{k_2}, \dots$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_{k_n} = f_*, \quad \mathbb{P} - a.e.$$

not only along the indicated subsequence, but also along ALL of its subsequence.

Hereditary behaviour.

We believe that a smilar result holds along the Hsu-Robbins-Erdös lines.

DESIDERATUM:

For any given sequence $f_1, f_2, \ldots$ of measurable functions, bounded in $\mathbb{L}^2$, i.e.,

$$\sup_{n \in \mathbb{N}} \mathbb{E}(f_n^2)] < \infty, \tag{1}$$

there exists an $f_* \in \mathbb{L}^2$ and a subsequence $f_{k_1}, f_{k_2}, \ldots$ such that

$$\sum_{N \in \mathbb{N}} \mathbb{P}(|\frac{1}{N} \sum_{n=1}^{N} f_{k_n} - f_* > \epsilon)| < \infty, \forall \epsilon > 0, \tag{2}$$

holds, not only along said subsequence, but also hereditarily.

<u>REMARK:</u>

Without sacrificing generality, for the purposes of establishing (2), the condition (1) can be replaced by the stronger condition

the sequence $(f_n^2)_{n=1}^{\infty}$ is uniformly integrable. (3).

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The sequence $f_1, f_2, \ldots$ contains a subsequence $f_{k_1}, f_{k_2}, \ldots$ whose squares converge weakly in $\mathbb{L}^1$ to a function $\eta \in \mathbb{L}^2$:

$$\lim_{n\to\infty} \mathbb{E}(f_{k_n}^2.\zeta) = \mathbb{E}(\eta.\zeta), \quad \forall \zeta \in \mathbb{L}^\infty.$$

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From the uniform integrability (3) and DUNFORD-PETTIS, we DO have the weak-$\mathbb{L}^1$ convergence (4) for some $\eta \in \mathbb{L}^1$. The real assumption here is

$$\eta \in \mathbb{L}^2.$$

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This DOES hold in a few important cases.

BOUNDEDNESS IN $\mathbb{L}^4$

Suppose

$$\sup_{n\in\mathbb{N}} \mathbb{E}(f_n^4) < \infty$$

holds.

This means that $f_1^2, f_2^2, \dots$ is bounded in $\mathbb{L}^2$. But then there exist a function $\eta \in \mathbb{L}^2$ and a subsequence $f_{k_1}, f_{k_2}, dots$ with

$$\lim_{n\to\infty} \mathbb{E}(f_{k_n}^2.\zeta) = \mathbb{E}(\eta.\zeta)$$

valid for every $\zeta \in \mathbb{L}^2$, thus also in $\mathbb{L}^\infty$.

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The hereditary aspect is automatic in the IID Case.

As we mentioned, the weak-$\mathbb{L}^1$ convergence

$$\lim_{n\to\infty} \mathbb{E}(f_{k_n}^2.\zeta) = \mathbb{E}(\eta.\zeta), \quad \forall \zeta \in \mathbb{L}^\infty.$$

can always be guaranteed along a suitable subsequence, and for some $\eta \in \mathbb{L}^1$.

INDEPENDENCE

As we mentioned, the weak-$\mathbb{L}^1$ convergence

$$\lim_{n\to\infty} \mathbb{E}(f_{k_n}^2.\zeta) = \mathbb{E}(\eta.\zeta), \quad \forall \zeta \in \mathbb{L}^\infty.$$

can always be guaranteed along a suitable subsequence, and for some $\eta \in \mathbb{L}^1$.

Now, if the $f_1, f_2, \ldots$ are independent, this η is a constant, and therefore trivially in $\mathbb{L}^2$.

IDEA OF PROOF.

Since the sequence $f_1, f_2, \ldots$ is bounded in $\mathbb{L}^2$, it contains a subsequence $f_{k_1}, f_{k_2}, \ldots$ which converges <u>weakly</u> in $\mathbb{L}^2$ to some $f_* \in \mathbb{L}^2$:

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Next Reduction: Assume the $f_{k_n}, \nu \in \mathbb{N}$ to be simple, and a martingale difference.

Thus

$$X_n \triangleq \sum_{n=1}^{N} f_{k_n}, \quad N \in \mathbb{N}$$

to be a square-integrable martingale. Now use a martingale theory for the job ...