On term structure modelling beyond multicurves

Wolfgang Runggaldier
(work in progress with Claudio Fontana and Simone Pavarana)

Dipartimento di Matematica, Università di Padova

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Describing the title

As possible post-crisis term structures beyond multi-curves

→ Attempts to consider a single driver for the various spreads *(Libor-OIS, Libors for different tenors, post-Libor spreads,....)*

One approach considers roll-over risk (Backwell et al, '21) *(shall see that it can be related to multiplicative spreads)*

- Our purpose here is to relate the *roll-over risk approach* to the risk attitude of a representative agent, which
  - i) does not require classical AOA *(using partly the Benchmark approach)*
  - ii) relates it to risk sensitive control.

Outline

- **Introductory notions**
  - Notions from the benchmark approach
  - Notions from Interest Rates
- Roll-over risk
- Risk-sensitive approach

Notions from the Benchmark Approach

Given is a diffusion-type market with

- $S_t = (S_t^1, \cdots, S_t^d)$ the risky asset prices, S_t^0 a riskless asset
- V_t^π : portfolio value at t for self-financing π
- θ_t : market price of risk

$\rightarrow$ if $\theta_t \in \mathcal{L}^2_{loc}$ then $\exists\,!$ the GOP V_t^* (also *numèraire portfolio*)

$\rightarrow$ *benchmarked prices: prices expressed in units of* V_t^*

Notions from the Benchmark Approach

When benchmarked prices are true martingales: fair pricing (*real world pricing) is possible which does not require classical AOA.*

- Next extend the original market by adding ZCBs and FRAs

$\rightarrow$ if ZCBs and FRAs can be fairly priced by V_t^*, then the original GOP remains such also in the extended market.

Recalling some notions from Interest Rates

- After the big crisis

$$L(t;T,T+\delta) \neq F(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{p(t,T)}{p(t,T+\delta)} - 1\right)$$

$L(t;T,T+\delta)$ $\downarrow$ forward Libor

$F(t;T,T+\delta)$ $\downarrow$ simply compounded forward rate

$\rightarrow$ Leads to spreads, e.g. the *multiplicative (Libor-OIS) spreads*

$$S(t;T,T+\delta) = \frac{1+\delta\, L(t;T,T+\delta)}{1+\delta\, F(t;T,T+\delta)}$$

$\rightarrow$ *Also with the current Libor reform there may still be analogous spreads.*

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Roll-over risk

Backwell, Macrina, Schlögl, Skovmand ('21) interpret interbank risk as roll-over risk: the risk of not being able to roll over the financing of debt at a given market reference rate (*at the same spread to a given market reference rate).*

→ *Roll-over risk may be due various risks, in particular credit risk, but mainly funding/liquidity risk and may thus be present also in absence of credit risk.*

- $A(t, T)$: value in $t < T$ of a loan of one monetary unit, continuously rolled over until T (*present value, in t, of the unsecured roll-over risk-adjusted borrowing account with maturity T).*

→ *One has $A(t, T) \geq 1$ and, if there is no roll-over risk, then $A(t, T) = 1$.*

Roll-over risk

Next we derive an explicit expression for $A(t, T)$ in terms of credit and funding spreads and assuming that it can be fairly priced by the GOP. In view of this:

Denote

$$\begin{aligned} \tau &: \text{default time of the counterparty of a loan} \\ \lambda_t &: \text{credit/downgrade spread} \\ \varphi_t &: \text{funding/liquidity spread} \\ \mathcal{G}_t &= \mathcal{F}_t \vee \mathcal{H}_t \quad (\mathcal{H}_t : \text{default history}) \end{aligned}$$

Roll-over risk

Assuming $A(t,T)$ to be fairly priced by the GOP and considering a funding account $\tilde{B}_t$ based on the extended interest rate process $\tilde{r}_t := r_t + \varphi_t$, we obtain

$$
\begin{aligned}
A(t,T) &= V_t^* \, E\left\{ \tfrac{1}{V_T^*} \exp\left[\int_t^T (r_s + \varphi_s + \lambda_s)\, ds \right] \mathbf{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right\} \\
&= V_t^* \, E\left\{ \tfrac{1}{V_T^*} \exp\left[\int_t^T (r_s + \varphi_s)\, ds \right] \mid \mathcal{F}_t \right\} := \frac{V_t^*}{\tilde{B}_t} \, E\left\{ \frac{\tilde{B}_T}{V_T^*} \mid \mathcal{F}_t \right\}
\end{aligned}
$$

- If there is roll-over risk, i.e. $A(t,T) > 1$ then one can easily see that $\tilde{B}_t$ cannot be fairly priced by V_T^* (the GOP).
- Below we shall present an alternative, where $\tilde{B}_t$ can be fairly priced and leading to a relation between the funding spread φ_t and a risk sensitivity parameter η of a representative agent.

Roll-over risk

In equilibrium the values in T of the term borrowing and roll-over borrowing over $[T, T+\delta]$ (δ a given tenor) must be equal. Assuming the defaultable bonds to coincide with the risk-free ones (*credit risk mitigated by collateralization*), this then leads to

$$(1+\delta\, L(T;T,T+\delta)\, p(T,T+\delta) = A(T,T+\delta)$$

i.e. we have

$$L(T;T,T+\delta) = \frac{1}{\delta}\left(\frac{A(T,T+\delta)}{p(T,T+\delta)} - 1\right)$$

$\rightarrow$ $A(t,T)$ can be seen as a single driver affecting the Libors differently according to the tenor.

Roll-over risk

For the multiplicative spread it then follows that

$$\begin{aligned} S(T;T,T+\delta) &= \frac{1+\delta\, L(T;T,T+\delta)}{1+\delta\, F(T;T,T+\delta)} \\ &= \frac{A(T,T+\delta)}{p(T,T+\delta)}\, p(T,T+\delta) = A(T,T+\delta) \end{aligned}$$

- $\rightarrow$ Since T and δ are arbitrary, identifying T with t and $T+\delta$ with T, we obtain the equality $S(t,T) := S(t;t,T) = A(t,T)$ (*links spreads to roll-over risk*).
- $\rightarrow$ The Libor-OIS (multiplicative) spread can thus also be seen as a premium paid by the borrower at Libor to avoid roll-over risk over the period of the loan.

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Risk sensitive approach

Next we want to obtain a description of $A(t,T)$ (and thus also of $S(t,T)$) in terms of the risk attitude of investors.

$\rightarrow$ It will provide a risk-sensitive interpretation of spreads as well as of the entire roll-over approach.

Dynamics

First we need some underlying dynamics for our market quantities. To this effect consider a factor process X_t satisfying

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dw_t$$

and assume it affects the dynamics in the market in the sense that

$$\begin{cases} dS_t &= diag(S_t)\,[\mu(t, X_t)\,dt + \sigma(t, X_t)\,dw_t] \\ dS^0_t &= S^0_t\, r(t, X_t)\,dt \qquad (w_t : global\ Wiener) \end{cases}$$

$\rightarrow$ Market price of risk

$$\theta(t, X_t) = \sigma^+(t, X_t)\,(\mu(t, X_t) - r(t, X_t)\,\underline{1})$$

(σ full rank $\Rightarrow$ $\sigma^+ = \sigma'(\sigma\sigma')^{-1}$)

$\rightarrow$ Let also the funding/liquidity spread be of the form $\varphi_t = \varphi(t, X_t)$.

Risk sensitive approach

- Consider a criterion to be maximized which is of the form

$$J^{\eta}(\pi) = -\frac{1}{\eta} \log\, E\left\{\exp\left[-\eta\, \log\, V_T^{\pi}\right]\right\}$$

with $\eta > -1$ and $\neq 0$ a risk sensitivity parameter.

- One may consider the reduced form

$$I^{\eta}(\pi) = E\left\{\exp\left[-\eta\, \log\, V_T^{\pi}\right]\right\} \quad \left(= E\left\{\left(V_T^{\pi}\right)^{-\eta}\right\}\right)$$

to be maximised/minimised depending on the sign of η.

→ The latter links risk-sensitive control to expected power utility (*can also be related to a mean-variance criterion*)

Risk sensitive approach

- Notice that fair pricing with the stochastic discount factor $1/V_t^*$ corresponds to a marginal utility pricing rule with $U(x) = \log(x) \quad (U'(V_t^*) = 1/V_t^*)$.

$\rightarrow$ Our $\tilde{B}_t$ was not correctly priced by logarithmic preferences since $U'(V_t^*)\,\tilde{B}_t$ resulted in a sub-martingale.

Next change to a more flexible utility $U(x) = x^{-\eta}$ (η *is as above a risk sensitivity parameter considered as referred to a representative agent*)

- Let π^η be a strategy resulting from the maximisation of $E\{(V_t^{\pi^\eta})^{-\eta}\}$ for a given risk sensitivity parameter η. The dynamics of $V_t^{\pi^\eta}$ can be appropriately derived.

Risk sensitive approach

As stochastic discount factor take now one associated with $U(x) = x^{-\eta}$, namely

$$Y_t := U'(V_t^{\pi^\eta}) = -\eta \left(V_t^{\pi^\eta}\right)^{-\eta-1}$$

Assumption: $\tilde{B}_t$ is correctly priced by the stochastic discount factor (SDF) Y_t, i.e. $Y_t \tilde{B}_t$ is a local martingale.

Risk sensitive approach

Working out the stochastic differential of $Y_t \tilde{B}_t$, a necessary and sufficient condition for it to be a local martingale is that

$$\varphi_t = \eta\, r + \theta'_t(\theta_t + \Xi_t) - \frac{2+\eta}{2\,(1+\eta)}\, \|\theta_t + \Xi_t\|^2$$

with Ξ_t a process (sometimes called *intertemporal hedging component*) resulting from the dynamics of $V_t^{\pi^\eta}$.

- $\rightarrow$ It connects φ with η
- $\rightarrow$ φ can be seen to be increasing with η
- $\rightarrow$ In the limit for $\eta \rightarrow 0$ (log-utility) we obtain $\varphi_t = 0$ (being also $\Xi_t = 0$) thus reconfirming that fair marginal utility pricing with log-utility implies that the funding spread φ_t has to be equal to zero..

Thank you Tomas