Modeling general default time under risk neutral probability

Conference in memory of Tomas Bjork

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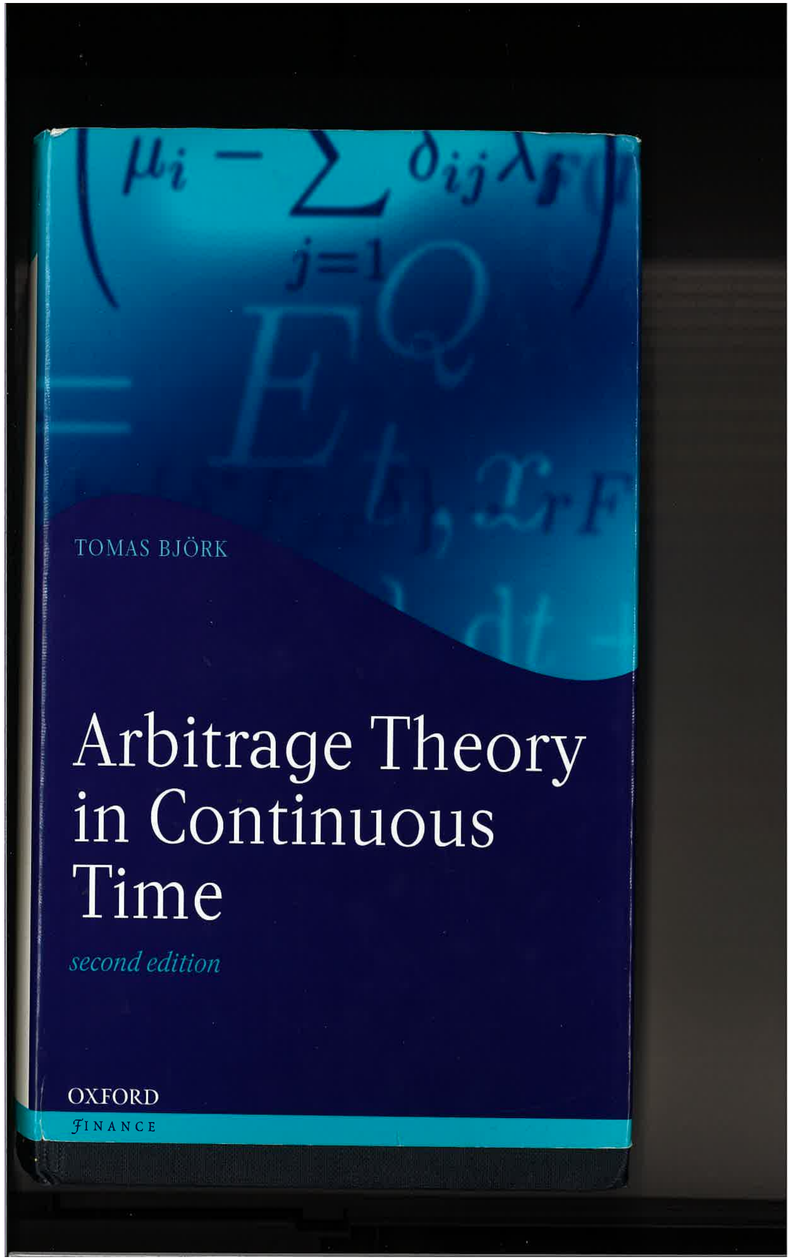
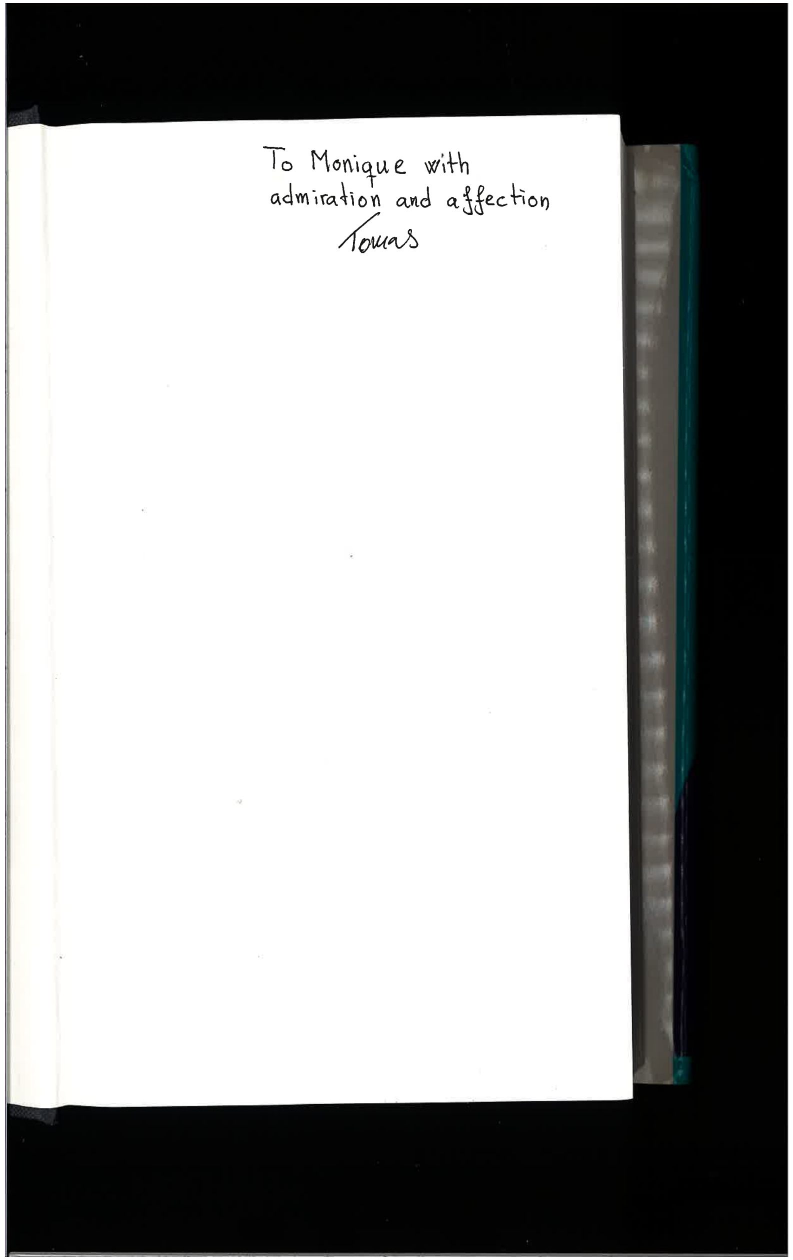
The first time I met Tomas was in 1992, in Oberwolfach. Then we met in various places: Trento, Bachelier workshops, QMF Sydney, Oberwolfach....

Risk and Stochastics Conference in honour of Professor Ragnar Norberg, 2015 April, LSE London.


TOMAS BJÖRK
Arbitrage Theory in Continuous Time
second edition
OXFORD
FINANCE


To Monique with
admiration and affection
Tomas

Generalities

We work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \widetilde{\mathbb{P}})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a complete and right-continuous filtration, $\mathcal{F}_0$ is trivial, and $\mathcal{G}$ is a σ-algebra satisfying $\mathcal{F}_\infty \subset \mathcal{G}$.

We are given a random time τ (a non negative random variable) defined on $(\Omega, \mathcal{G})$, with law denoted by η and we assume that η is non-atomic. We introduce

$$A_t = 1\!\!1_{\{\tau \leq t\}}, \forall t \geq 0$$

and we denote by $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ the smallest filtration containing $\mathbb{F}$ and turning out τ into a $\mathbb{G}$-stopping time: if $\mathbb{A}$ is the filtration generated by the process A, $\mathbb{G} = \mathbb{F} \vee \mathbb{A}$.

We denote by Z the càdlàg **Azéma supermartingale** associated with τ, which is the optional projection of $1 - A$ and satisfies

$$Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \forall t \geq 0,$$

and the optional Azéma supermartingale $\widetilde{Z}$ which is the optional projection of $1 - A_-$, and satisfies

$$\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t), \forall t \geq 0\,.$$

The **Doob-Meyer decomposition** of Z is $Z = M - A^p$, where M is a martingale and A^p the dual predictable projection of A.

The *compensator* of A is the $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$ such that $A - \Lambda^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. There exists an $\mathbb{F}$-predictable increasing process Λ such that $\Lambda^{\mathbb{G}}_t = \Lambda_{t\wedge\tau}, \forall t \geq 0$. If $Z_- > 0$

$$d\Lambda_t = \frac{dA^p_t}{Z_{t-}}, \, \forall t \geq 0, \, \Lambda_0 = 0 \, .$$

The $(\mathbb{P}, \mathbb{F})$-**conditional cumulative function** of τ is defined, for any $(t, u) \in \mathbb{R}_+^2$ by

$$F_t(u) = \mathbb{P}(\tau \leq u | \mathcal{F}_t) \, .$$

The family $(F(u), u \in \mathbb{R}_+)$ is a family of $(\mathbb{P}, \mathbb{F})$-martingales, valued in $[0, 1]$, increasing with respect to the parameter u (i.e., $F_t(u) \leq F_t(v), a.s.$ for $u < v, \forall t \geq 0$).

Conditional densities

The random time τ admits a $\mathbb{P}$-conditional density in the sense of Jacod (we shall say a J-conditional density) if there exist a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $(\omega, t, u) \to p_t(\omega, u)$ càdlàg in t such that
(J1) for every u, the process $(p_t(u))_{t\geq 0}$ is a non-negative $\mathbb{F}$-martingale,
(J2) for every $t \geq 0$, the measure $p_t(u)\eta(du)$ equals $\mathbb{P}(\tau \in du \,|\, \mathcal{F}_t)$, in other words, for any Borel bounded function h, for any $t \geq 0$

$$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)p_t(u)\eta(du)\,.$$

Generalized density

The random time τ admits a generalized density in the sense of Jiao & Li (we shall say a JL-conditional density) if there exist a increasing family $(\tau_i, i \geq 1)$ with no accumulation point of $\mathbb{F}$-stopping times and a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $(\omega, t, u) \to \alpha_t(\omega, u)$ càdlàg in t such that
(JL1) for every u, the process $(\alpha_t(u))_{t \geq 0}$ is a non-negative $\mathbb{F}$-martingale,
(JL2) for any bounded Borel function h and $t \geq 0$,

$$\mathbb{E}[h(\tau) \prod_{i=1} \mathbb{1}_{\{\tau \neq \tau_i\}} | \mathcal{F}_t] = \int_{\mathbb{R}_+} h(u) \alpha_t(u) \pi(du)$$

where π is a non-negative, non atomic measure on $\mathbb{R}_+$.

Immersion

If $\mathbb{F}$ and $\mathbb{K}$ are two nested filtrations ($\mathbb{F} \subset \mathbb{K}$), immersion holds under $\mathbb{P}$ if any $\mathbb{F}$-local $\mathbb{P}$-martingale is a $(\mathbb{P}, \mathbb{K})$-local martingale.

If $\mathbb{F} \subset \mathbb{G} \subset \mathbb{K}$ and $\mathbb{F}$ is immersed in $\mathbb{K}$, then $\mathbb{F}$ is immersed in $\mathbb{G}$.

If $\mathbb{K} = \mathbb{F} \vee \sigma(\Theta)$ where Θ is independent of $\mathbb{F}$ under $\mathbb{P}$, immersion holds under $\mathbb{P}$.

Financial market

We assume that S is the price of a traded asset, $\mathbb{F}$-adapted, and that the financial market (with a riskless asset with zero interest rate) $(S, \mathbb{F}, \widetilde{\mathbb{P}})$ satisfies NFLVR assumption.

We assume that the financial market $(S, \mathbb{G}, \widetilde{\mathbb{P}})$ (where we use $\mathbb{G}$-predictable portfolio) satisfies NFLVR assumption.

If the $(S, \mathbb{F}, \widetilde{\mathbb{P}})$ market is complete, immersion holds under any e.m.m. in the enlarged market.

In the incomplete case, under some conditions, there exists at least a probability measure $\mathbb{P}$ on $\mathbb{G}$ such that $\mathbb{P}$ is equivalent to $\widetilde{\mathbb{P}}$ and immersion holds under $\mathbb{P}$.

Under immersion, Z is **decreasing**. This is why we study that case.

Pricing:

In order to price defaultable claims one needs to know $Z, \widetilde{Z}, A^o$ and A^p.

Generalized Cox model

We now assume that the σ-algebra $\mathcal{G}$ is large enough to support a random variable Θ with unit exponential law, independent from $\mathcal{F}_\infty$. Lando has introduced the Cox model

$$\tau = \inf\{t \geq 0 \,:\, \int_0^t \lambda_s ds \geq \Theta\},$$

where λ is a non negative $\mathbb{F}$-adapted process.

We study the generalized Cox model where K is an increasing càdlàg $\mathbb{F}$-adapted process such that $K_0 = 0$. We define

$$\tau = \inf\{t \geq 0 \,:\, K_t \geq \Theta\}.$$

Then,

$$Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(K_t < \Theta | \mathcal{F}_t) = e^{-K_t}\,, \quad \widetilde{Z}_t = Z_{t-}\,.$$

Immersion holds under $\mathbb{P}$ between the reference filtration $\mathbb{F}$ and $\mathbb{G}$, its progressive enlargement with τ, since, obviously $\mathbb{F}$ is immersed in $\mathbb{F} \vee \sigma(\Theta)$ and $\mathbb{F} \subset \mathbb{G} \subset \mathbb{F} \vee \sigma(\Theta)$.

Under immersion, $1 - Z = A^o = {}^oA$, where ${}^oA$ is the optional projection of A and A^o the dual optional projection of A, and $\widetilde{Z} = 1 - A^o_-$.

The conditional survival process is

$$\begin{aligned} \mathbb{P}(\tau > u | \mathcal{F}_t) &= Z_u, \quad \text{for } u < t \\ &= \mathbb{E}[Z_u | \mathcal{F}_t], \quad \text{for } t \leq u\,. \end{aligned}$$

The case where K is continuous

In that case, τ avoids all $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta < \infty) = 0$ for any $\mathbb{F}$-stopping time ϑ, and, in particular, A^o being continuous, $\widetilde{Z} = Z$ and $A^p = A^o$.

Moreover, if K is absolutely continuous with respect to the law of τ, i.e., $K_t = \int_0^t k_u \eta(du)$, $\forall t \geq 0$, then, the J-conditional density is given by :

$$\begin{aligned} p_t(u) &= k_u e^{-K_u}, \quad \text{for}\, u < t, \\ &= \mathbb{E}[k_u e^{-K_u} | \mathcal{F}_t], \quad \text{for}\, t \leq u\,. \end{aligned}$$

If K is continuous but not absolutely continuous, the J-density may fail to exist, as we show now.

Let K be the continuous increasing process defined by
$K_t = -\ln(1 - L_{t\wedge 1}) + 1\!\!1_{\{t>1\}}(t-1)$, $t \geq 0$, where L is the local time at level 0 of a standard Brownian motion. Then $\mathbb{P}(\tau > t) = \mathbb{E}[1 - L_{t\wedge 1}]e^{(t-1)^+}$ and, from $\mathbb{E}[L_t] = \mathbb{E}[|W_t|] = \frac{\sqrt{2t}}{\sqrt{\pi}}$, we deduce that τ has a density f with respect to Lebesgue's measure. Therefore, if the J-conditional hypothesis is satisfied then, one would have, for $u < t$

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = Z_u = \int_u^\infty p_t(s)f(s)ds$$

and Z would be absolutely continuous with respect to Lebesgue's measure, which is not the case. The processs K being continuous, τ avoids $\mathbb{F}$-stopping times and the J-density does not exists.

The case where K is càdlàg

Let K be an increasing $\mathbb{F}$-adapted càdlàg process, with $K_0 = 0$, $K_t < \infty$ for all $t \geq 0$ and $K_\infty = \infty$. We assume that the sequence of jump times is strictly increasing, without accumulation points.

For ω fixed, the set $\{t : K_t \geq \Theta\}$ is of the form $[t_0, \infty[$ with $K_{t_0} \geq \Theta$ and $\tau = t_0$, hence

$$\{K_t < \Theta\} = \{\tau > t\}.$$

For any finite $\mathbb{F}$-stopping time ϑ, one has $\mathbb{P}(\tau = \vartheta | \mathcal{F}_t) = \mathbb{E}[e^{-K_{\vartheta -}} - e^{-K_\vartheta} | \mathcal{F}_t]$. In particular,

$$\mathbb{P}(\tau = \vartheta < \infty) = \mathbb{E}[e^{-K_{\vartheta -}} - e^{-K_\vartheta}],$$

hence, τ avoids all $\mathbb{F}$-stopping times if and only if K is continuous. The $\mathbb{F}$-stopping times which are not avoided by τ are the jump times of K.

Doob-Meyer decomposition of Z

From stochastic calculus $dZ_t = -e^{-K_{t-}}(dK_t^c + dI_t)$ where K^c is the continuous part of the increasing process K and I is the special $\mathbb{F}$-semimartingale $(I_t = \sum_{s \leq t}(1 - e^{-\Delta K_s}), t \geq 0)$.

Being bounded Z is of class (D) and admits a Doob-Meyer decomposition.
The special $\mathbb{F}$-semimartingale I admits a canonical decomposition $I = M^I + A^I$, with A^I a predictable finite variation process and M^I a local martingale. Then, $Z = M - A^p$ where $dM_t = e^{-K_{t-}} dM_t^I$ and

$$dA_t^p = e^{-K_{t-}}(dK_t^c + dA_t^I), \; A_0^p = 0\,.$$

Conditional densities

If the continuous part of K is absolutely continuous with respect to a non negative non atomic measure π, i.e., $K^c_t = \int_0^t k_u \pi(du)$, the JL-conditional density exists with $\alpha_t(u) = \mathbb{E}[k_u e^{-K_u} | \mathcal{F}_t]$, $u \geq t$.

Let us point out a "technical" difficulty. Assume that K has no jumps at constant time, i.e. $\mathbb{P}(\tau = t) = 0, \forall t > 0$. This does not imply that $Z = \widetilde{Z}$ (the equality meaning that the two processes are indistinguishable). Indeed, $Z = 1 - A^o$ whereas $\widetilde{Z} = 1 - A^o_-$.

Other computations Let $(\tau_i)_{i\geq 1}$ be the sequence of jump times of K. The conditional probability that the default occurs at time τ_i is

$$p^i_t := \mathbb{P}(\tau = \tau_i|\mathcal{F}_t) \quad = \quad \mathbb{E}\left[e^{-K_{\tau_i -}}(1-e^{-\Delta K_{\tau_i}})|\mathcal{F}_t\right], \forall t \geq 0\,.$$

Note that $p^i_t = p^i_{t\wedge\tau_i}$, a result due to immersion. This implies that $p^i_{t\vee\tau_i} = p^i_{\tau_i}$.

For a bounded Borel function h, if the JL-conditional density exists,

$$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_0^\infty h(u)\alpha_t(u)\pi(du) + \sum_{i\geq 1}\mathbb{E}[h(\tau_i)p^i_{\tau_i}|\mathcal{F}_t], \forall t \geq 0\,.$$

The conditional survival probability is given by

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) \quad &= \quad e^{-K_u}, \quad \text{for } \ t \geq u \\ &= \quad \int_u^\infty \alpha_t(y)\pi(dy) + \sum_{i\geq 1}\mathbb{E}[\mathbb{1}_{\{\tau_i > u\}}p^i_{\tau_i}|\mathcal{F}_t], \quad \text{for } \ t < u\,.\end{aligned}$$

Examples of càdlàg processes K

Example 1

We can generalize the model of Jiao & Li.

We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, a continuous increasing $\mathbb{F}$-adapted process Γ and $(\tau_i)_{i\geq 1}$ a strictly increasing sequence with no accumulation points of $\mathbb{F}$-stopping times and we set $\tau_0 = 0$. We introduce the process A^i setting $A^i_t = 1\!\!1_{\{\tau_i \leq t\}}, \forall t \geq 0$, and its $\mathbb{F}$-compensator J^i. One denotes by Ψ an increasing deterministic function such that $\Psi(0) = 0,\ \ \Psi(t) < \infty, \forall t \geq 0$ and $\Psi(\infty) = \infty$. We set

$$K_t = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}} [\Psi(\tau_i) - \Psi(\tau_{i-1})] + \Gamma_t, \forall t \geq 0\,.$$

Then,

$$A^p_t = \Gamma_t + \sum_{i\geq 1} \int_0^t \left(e^{-\Psi(\tau_{i-1})} - e^{-\Psi(s)}\right) dJ^i_s,\ \forall t \geq 0\,.$$

Example 2

If a strictly increasing sequence $(\tau_i)_{i\geq 1}$ with no accumulation points of $\mathbb{F}$-stopping times and a sequence of non-negative random variables $(\theta_i \in \mathcal{F}_{\tau_i}, i \geq 1)$ are given, as well as a non-negative $\mathbb{F}$-adapted process k, one sets

$$K_t = \int_0^t k_s ds + \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}} \theta_i \,.$$

The associated random time τ does not avoid the $\mathbb{F}$-stopping random times $(\tau_i)_{i\geq 1}$ and we set $\tau_0 = 0$. We have

$$I_t = \sum_{i\geq 1} (1 - e^{-\theta_i}) 1\!\!1_{\{\tau_i \leq t\}} = \sum_{i\geq 1} \gamma_i 1\!\!1_{\{\tau_i \leq t\}} = M_t^I + A_t^I \,.$$

We consider the marked point process $(\gamma_i, \tau_i)_{i\geq 1}$ with jump measure μ defined as

$$\mu(\omega, [0,t], A) = \sum_{i\geq 1} 1\!\!1_{\{\tau_i(\omega) \leq t\}} 1\!\!1_{\{\gamma_i(\omega) \in A\}}, \forall t \geq 0$$

and its compensator ν.

Then $M^I = \int_0^\cdot \int_{\mathbb{R}_+} x(\mu(ds, dx) - \nu(ds, dx))$ and $A^I = \int_0^\cdot \int_{\mathbb{R}_+} x\nu(ds, dx)$.

Furthermore, $dA^p = e^{-K_-}(dK^c + dA^I)$, and $d\Lambda = dK^c + dA^I$.

Example 3: Shotnoise

We are given a sequence (τ_i) of $\mathbb{F}$-stopping times and a family (γ_i) of $\mathcal{F}_{\tau_i}$-measurable random variables and we set

$$K_t = \sum_{i \geq 1} 1\!\!1_{\{\tau_i \leq t\}} G(t - \tau_i, \gamma_i) = \int_0^t \int_{\mathbb{R}} G(t-s, x)\mu(ds, dx), \forall t \geq 0\,,$$

where G is a function $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$.

Note that $\Delta K_{\tau_i} = G(0, \gamma_i)$ and $K_t^c = \sum_{i \geq 1} 1\!\!1_{\{\tau_i \leq t\}} [G(t - \tau_i, \gamma_i) - G(0, \gamma_i)], \forall t \geq 0.$

We assume that

$$G(t, x) = G(0, x) + \int_0^t g(s, x) ds, \quad \forall t \geq 0,\ x \in \mathbb{R},$$

where g is a non-negative Borel function on $\mathbb{R}_+ \times \mathbb{R}$, so that G is increasing with respect to its first variable.

Under other technical hypotheses, the Doob-Meyer decomposition of Z is $Z = M - A^p$ with

$$
\begin{aligned}
M_t &= \int_0^t Z_{s-} \int_{\mathbb{R}} (e^{-G(0,x)} - 1)\widetilde{\mu}(ds, dx) \\
A^p_t &= \int_{u=0}^t Z_u k_u du + \int_0^t \int_{\mathbb{R}} (e^{-G(0,x)} - 1)\nu(ds, dx)\,,
\end{aligned}
$$

where $k_u = \int_{\mathbb{R}} \int_{s=0}^u g(u-s,x)\, \mu(ds,dx)$.

Pricing of defaultable claims

It is then easy to price defaultable claims using $\mathbb{P}$ as the pricing measure and obtain closed form solutions and see the impact of the no-avoidance of $\mathbb{F}$ stopping times which, in general, produces jumps in these prices.

As an example, we consider a zero coupon defaultable bond with maturity T, which delivers one monetary unit at maturity if and only if the default did not occur before T. We assume that the riskless interest rate is null. The price of the defaultable bond, if $\mathbb{P}$ is the pricing measure, is

$$D_t(T) = \mathbb{P}(\tau > T|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \frac{\mathbb{E}[Z_T|\mathcal{F}_t]}{Z_t} = 1\!\!1_{\{t<\tau\}} \widetilde{D}_t(T)$$

where $\widetilde{D}(T)$ is the predefault price $\widetilde{D}_t(T) = \mathbb{E}[Z_T|\mathcal{F}_t]e^{K_t}$.

We assume to be in the case of shot-noise model where ν is deterministic. It follows, with simple computations that,

$$\widetilde{D}_t(T) = \exp\Big(\int_t^T \int_{\mathbb{R}} (e^{-G(T-s,x)} - 1)\nu(ds, dx) - \int_0^t \int_{\mathbb{R}} [G(T-s, x) - G(t-s, x)]\mu(ds, dx) \Big) .$$

In particular, $\widetilde{D}$ has jumps at times $(\tau_i)_{i\geq 1}$ with negative jump sizes :

$$\Delta \widetilde{D}_{\tau_i} = \widetilde{D}_{\tau_i -}(e^{-(G(T-\tau_i,\gamma_i)-G(0,\gamma_i))} - 1) .$$

Note also that $\tilde{D}(T)$ is increasing between two jumps.

The dynamics of $\widetilde{D}$ can be obtained

$$\begin{aligned} d\widetilde{D}_t(T) &= \widetilde{D}_{t-}(T)\int_{\mathbb{R}}(e^{G(0,x)-G(T-t,x)}-1)\widetilde{\mu}(dt,dx) \\ &\quad +\widetilde{D}_{t-}(T)\Big(dA^J_t+\int_{\mathbb{R}}(e^{-G(T-t,x)}-1)(e^{G(0,x)}-1)\nu(dt,dx)\Big)\,, \end{aligned}$$

where

$$A^J_t=\int_{u=0}^t\Big(\int_{\mathbb{R}}\int_{s=0}^u h^J(u-s,x)\;\mu(ds,dx)\Big)du+\int_0^t\int_{\mathbb{R}}H^J(0,x)\nu(ds,dx)\,,$$

where $h^J(t,x)=g(t,x)e^{G(t,x)}$ and $H^J(0,x)=e^{G(0,x)}-1$

On going work

It remains to produce numerics and calibration. The case of multiple defaults is a work in progress.

References

[1] Aksamit, A. and Jeanblanc, M., *Enlargement of filtration with finance in view*, Springer, 2017.

[2] Aksamit, A. and Li, L., Projections, pseudo-stopping times and the immersion property, *Séminaire de Probabilités XLVIII*, Lecture Notes in Mathematics, Edts Donati-Martin, C. and Lejay, A. and Rouault, A., 187-218, 2016.

[3] Bélanger, A., Shreve, S. E. and Wong, D. A general framework for pricing credit risk,*Mathematical Finance*, 14,3, 317–350, 2004.

[4] Gueye, D. and Jeanblanc, M. Generalized Cox Model for Default Times, forthcoming in *Mathematics Going Forward,* Lecture Notes in Mathematics 2313, ch. 9, Springer.

[5] Gehmlich, F. and Schmidt, Th. Dynamic Defaultable Term Structure Modeling Beyond the Intensity Paradigm, *Mathematical Finance*, 2016.

[6] Jeanblanc, M. and Le Cam, Y. Progressive enlargement of filtrations with initial times. *Stochastic Processes and their Applications* **119**, 2523–2543, 2009.

[7] Jiao, Y. and Li, S., Generalized density approach in progressive enlargement of filtrations, *Electronic Journal of Probability*, 20, 2015.

[8] Jiao, Y. and Li, S. Modeling Sovereign Risks: From a Hybrid Model to the Generalized Density Approach, *Mathematical Finance*, 2016.

Thank you for your attention

The predictable projection of X is the unique predictable process ${}^pX$ such that for any $\mathbb{F}$-predictable stopping time ϑ

$$\mathbb{E}(X_\vartheta 1\!\!1_{\{\vartheta<\infty\}}) = \mathbb{E}({}^pX_\vartheta 1\!\!1_{\{\vartheta<\infty\}})\,.$$

Let $(X_t,\, t \geq 0)$ be an integrable increasing process. There exists a unique integrable $\mathbb{F}$-predictable increasing process $(X_t^p, t \geq 0)$, called the dual predictable projection of X such that, for any $\mathbb{F}$-predictable process Y,

$$\mathbb{E}\left(\int_0^\infty Y_s dX_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dX_s^{(p)}\right)$$