Mean field games in Itô diffusion markets under general preferences

Workshop in memory of Tomas Björk

Thaleia Zariphopoulou
The University of Texas at Austin
Oxford-Man Institute, Oxford University

My personal encounters with Tomas

1. Meeting at Strobl, 1999

The joy of discovering a deeper meaning through an alternative proof

2. Visit to UT-Austin, 2006

The term structure of tee times

3. Summer and Fall 2020

Finishing the book on jump processes amid very challenging times

Tomas,

stort tack för allt du gav till oss

och till finansmatematiken.

This talk

- **Mean field games in Itô diffusion markets**

  Controls in both drift and volatility

  General risk preferences

  Common noise

- **Forward performance criteria**

  Itô diffusion markets

  Related SPDE

Competition in portfolio choice

Performance is frequently viewed in relative sense

- Prevalent in mutual and hedge fund management
- Also present in various investment problems with benchmarks or maintenance of standards of living

Modeling competition

- Asset specialization
- Asset diversification

Competition among fund managers

Chevalier and Ellison (1997)
Sirri and Tufano (1998)
Agarwal, Daniel and Naik (2004)
Ding, Getmansky, Liang and Wermers (2007)
Goriaev et al. (2003)
Li and Tiwari (2006)
Gallaher, Kaniel and Starks (2006)
Brown, Goetzmann and Park (2001)
Kempf and Ruenzi (2008)
Basak and Makarov (2013, 2016)
Espinosa and Touzi (2015), ...

Career advancement motives, seeking higher money inflows from their clients, preferential compensation contracts, ...

Only two managers, mainly discrete-time models, criteria involving risk neutrality, relative performance with respect to an absolute benchmark or a critical threshold, constraints on the managers' risk aversion parameters

Modeling competition

- Best-response strategies
- Nash equilibrium

Stochastic optimization models

- Multi-dim. expected utility problems of common, across competitors, horizon and with payoffs involving relative quantities
- In general, tractable **only** for special utilities and market environments

References

Anthropelos, Geng and Z. (2022)
Espinosa and Touzi (2015)
Fu, Su and Zhou (2020)
Geng (2016)
Guo, Xu and Z. (2020)
Huang and Nguyen (2016)
Hu and Z. (2021)
Kraft, Meyer-Wehmann and Seifried (2020)
Lacker and Soret (2020)
Lacker and Z. (2019)
Reis and Platonov (2020, 2021)
Whitmeyer (2019)
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Main ingredients of models in classical setting

- Horizon is pre-assigned and common
- Utility is pre-assigned
- Market model is pre-assigned

- In the best-response setting, the policies of the competitors are assumed to be entirely known

These assumptions are, however, frequently contradicted in many empirical studies

Challenging the existing assumptions

- There is no single-horizon as such, as there is "rolling" from one performance period to the next
- Horizon might not be common, competitors might have their own internal horizons
- Managers adjust their targets depending on past realized performance, unpredictable events, etc.
- Model decay frequently occurs, especially when many assets are involved
- Competitors' strategies are not always a priori known

References

Anthropelos et al. (2022), Huang, Sialm and Zhang (2011), Kempf, Ruenzi and Thiele (2009), Dong, Feng and Sadka (2019), Barber, Huang and Odean (2016), Lynch and Musto (2003), Bodnaruk and Simonov (2016)

Forward performance criteria (Musiela–Z. 2002)

Give rise to ill-posed problems

Have direct connections with:

- Random HJB equations
- SPDE
- Ergodic control
- Ergodic BSDE
- Infinite horizon BSDE
- Martin boundary

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Many interesting mathematical problems remain open

El Karoui, Mrad, Hillairet, Nadtochiy, Henderson, Hobson, Chong, Liang, Sircar, Tehranchi, Rogers, Strub, Angoshtari, Zhou, Zitkovic, Anthropelos, Reis, . . .

Forward performance criteria

- **Axiomatic foundation** (in progress)

  Joint project with Nicole El Karoui and Mohamed Mrad

- **Volume on "*Recent developments on forward performance criteria*"**

  Co-editing with G. Liang

  Probability, Uncertainty and Quantitative Risk

N-player games in Itô diffusion markets

Stochastic market environment and the players

- **Market**: a riskless security and a stock

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t, \qquad S_0 > 0$$

  W is a standard Brownian motion in $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \left\{\mathcal{F}_t^W\right\}_{t \geq 0}$

  Market coefficients μ_t, $\sigma_t \in \mathcal{F}_t^W$, $\sigma_t > 0$ and $\left|\frac{\mu_t}{\sigma_t}\right| \leq c(t)$, $t \geq 0$

- N-**players** invest in both assets using self-financing policies $\pi_i \in \mathcal{A}$

$$\mathcal{A} := \left\{\pi : \pi_t \in \mathcal{F}_t^W \text{ and } E_{\mathbb{P}} \int_0^t \sigma_s^2 \pi_s^2 \,\mathrm{d}s < \infty\right\}$$

  Their wealth processes $X_{i,t}$, $t \geq 0$, $i = 1, \ldots, N$, solve

$$\mathrm{d}X_{i,t}^{\pi_i} = \pi_{i,t}(\mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t), \qquad X_{i,0} = x_i \in \mathbb{R}$$

- Players **compete** with each other in that their own assessment of how well they do is relative to the performance of their competitors

Competition

Forward best-response criteria

Forward best-response criteria

Competitors' policies: $(\pi_{-i,t}) := (\pi_{1,t}, \ldots, \pi_{i-1,t}, \pi_{i+1,t}, \ldots, \pi_{N,t}) \in \mathcal{A}$

Forward best-response criteria

An $\mathcal{F}_t^W$-adapted process $U_i(x_1, \ldots, x_N, t; (\pi_{-i}))$, $t \geq 0$, is called a best-response forward relative performance criterion for player i if:

- For each $t \geq 0$, $x_{-i} \in \mathbb{R}^{N-1}$, $U_i(x_i, x_{-i}, t; (\pi_{-i}))$ is strictly increasing and concave in x_i, a.s.

- For each $\pi_i \in \mathcal{A}$, $U_i(X_{1,t}^{\pi_1}, \ldots, X_{i,t}^{\pi_i}, \ldots, X_{N,t}^{\pi_N}, t; (\pi_{-i}))$, $t \geq 0$, is a (local) supermartingale for all $\pi_j \in \mathcal{A}$, $j \neq i$

- There exists $\pi_i^* \in \mathcal{A}$, such that $U_i(X_{1,t}^{\pi_1}, \ldots, X_{i,t}^{\pi_i^*}, \ldots, X_{N,t}^{\pi_N}, t; (\pi_{-i}))$, $t \geq 0$, is a (local) martingale for all $\pi_j \in \mathcal{A}$, $j \neq i$

No requirement to a priori know the competitors' policies

Forward competition under linear wealth distortions

Competition under linear wealth distortions

Recall existing criteria in the **classical setting** (Espinosa and Touzi (2015), Lacker and Z. (2019), . . .)

$$v_i\left(x_1,\dots,x_N,t;(\pi_{-i})_{t\le s\le T}\right)$$

$$=\sup_{\pi_i} E_{\mathbb{P}}\left[V_i\left(X^{\pi_i}_{i,T}-\frac{\theta_i}{N}\sum_{j=1}^{N}X^{\pi_j}_{j,T}\right)\middle| X_{1,t}=x_1,\dots,X_{N,t}=x_N\right]$$

Important observations

- The value function process is a **special case** of a forward criterion
- $V_i\in\mathcal{F}^W_0$ and **not** $\mathcal{F}^W_T$
- The policies $(\pi_{-i})_{t\le s\le T}$ of all competitors must be **known for the entire horizon** $[t,T]$, otherwise the above problem cannot be solved

Competition under linear wealth distortions in the forward framework

Individual wealth with competition

$$X_t^{\pi_i} \quad \rightarrow \quad X_t^{\pi_i} - \frac{\theta_i}{N} \sum_{j=1}^{N} X_{j,t}^{\pi_j}, \qquad t \geq 0, \quad \theta_i \in (0,1)$$

Distorted initial utility

$$u_{i,0}(x_i) \ \rightarrow \ u_{i,0}(x_i - \theta_i \hat{x}) \qquad \hat{x} = \frac{1}{N} \sum_{j=1}^{N} x_j$$

Solution to forward best-response criterion (Z. 2022)

$$U_i(x_1, \ldots, x_N, t; (\pi_{-i})) = u_{i,0}\left(x_i - \theta_i \hat{x}, \int_0^t \lambda_s^2 \, \mathrm{d}s\right)$$

where $u_{i,0}(x,t)$ solves

$$u_{i,t} = \frac{1}{2}\frac{u_{i,x}^2}{u_{i,xx}} \qquad \text{with} \qquad \left(u'_{i,0}\right)^{-1}(x,0) = \int_{\mathbb{R}} \frac{x^{-y}-1}{y}\nu_i(\mathrm{d}y)$$

- The forward process is aligned with the evolution of the market price of risk λ_t
- Input from each player is expressed through the personalized measures ν_i
- If ν_i is known then $u_i(x,t)$ can be uniquely specified

Optimal control policy of i^{th} player

$$\pi_{i,t}^{*,x_i} = \frac{1}{1-\frac{\theta_i}{N}} \pi_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{\theta_i}{N}} \frac{1}{N} \sum_{j\neq i} \pi_{j,t}^{x_j}$$

$$= \frac{\lambda_t}{\sigma_t} \frac{1}{1-\frac{\theta_i}{N}} h_{i,x} \left(h_i^{(-1)}(x_i - \theta_i\hat{x}, 0) + \int_0^t \lambda_s^2 \,\mathrm{d}s + \int_0^t \lambda_s \,\mathrm{d}W_s, \int_0^t \lambda_s^2 \,\mathrm{d}s \right)$$

$$+ \frac{\theta_i}{1-\frac{\theta_i}{N}} \frac{1}{N} \sum_{j\neq i} \pi_{j,t}^{x_j}$$

- Effects of competition is present through the initial endowment $x_i - \theta_i\hat{x}$
- No information on the entire policies $(\pi_{-i,t})_{t\geq 0}$ is needed; this is not the case in the traditional setting in which we need to preassume $(\pi_{-i,t})_{0\leq t\leq T}$

Optimal forward wealth of the i^{th} player

$$X_{i,t}^{*,x_i} = \frac{1}{1-\frac{\theta_i}{N}} \left(z_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{N} \sum_{j\neq i} X_{j,t}^{\pi_j,x_j} \right)$$

$$= \frac{1}{1-\frac{\theta_i}{N}} h_{i,x} \left(h_i^{(-1)}(x_i - \theta_i\hat{x}, 0) + \int_0^t \lambda_s^2 \,\mathrm{d}s + \int_0^t \lambda_s \,\mathrm{d}W_s, \int_0^t \lambda_s^2 \,\mathrm{d}s \right)$$

$$+ \frac{\theta_i}{1-\frac{\theta_i}{N}} \frac{1}{N} \sum_{j\neq i} X_{j,t}^{\pi_j,x_j}$$

Despite the full nonlinearity of the dynamics, the first part of $X_{i,t}^{*,x_i}$ depends only on $x_i - \theta_i\hat{x}$ and the model dynamics

Example

Let $\nu_i(\mathrm{d}y) = \delta_0$, $i = 1, \ldots, N$. Then, $h_i(x_i, t) = x_i$ and $u_i(x, t) = 1 - e^{-x_i + \frac{t}{2}}$

Forward best-response criterion

$$U_i(x_1, \ldots, x_N, t) = 1 - \exp\left(-(x_i - \theta_i \hat{x}) + \frac{1}{2}\int_0^t \lambda_s^2 \, \mathrm{d}s\right)$$

Comparison with the classical case (Hu and Z. 2021)

$$v_i(x_1, \ldots, x_N, t) = 1 - \exp\left(-(x_i - \theta_i \hat{x}) + \frac{1}{2} E_{\mathbb{Q}_{[0,T]}}\left[\int_t^T \lambda_s^2 \, \mathrm{d}s \,\middle|\, \mathcal{F}_t^W\right]\right)$$

Note the need to fully prespecify the market model in the entire $[0, T]$

Example (cont'd)

Forward best-response optimal policies

$$\pi^*_{i,t} = \frac{1}{1-\frac{\theta_i}{N}}\frac{\lambda_t}{\sigma_t} + \frac{\theta_i}{1-\frac{\theta_i}{N}}\frac{1}{N}\sum_{j\neq i}\pi_{j,t}$$

Forward best-response controlled state processes

$$X^{*,x_i}_{i,t} = \frac{1}{1-\frac{\theta_i}{N}}\left(x_i + \int_0^t \lambda_s^2\,\mathrm{d}s + \int_0^t \lambda_s\,\mathrm{d}W_s + \frac{\theta_i}{N}\sum_{j\neq i} X^{\pi_j,x_j}_{j,t}\right)$$

Forward Nash equilibrium

Forward Nash equilibrium

A forward Nash equilibrium consists of N pairs of $\mathcal{F}_t^W$-adapted processes,

$$\left\{\left(U_1(x_1,\ldots,x_N,t),\pi_{1,t}^*\right),\ldots,\left(U_i(x_1,\ldots,x_N,t),\pi_{i,t}^*\right),\right.$$
$$\left.\ldots,\left(U_N(x_1,\ldots,x_N,t),\pi_{N,t}^*\right),\ t\geq 0\right\}$$

with the following properties:

i) For $t \geq 0$, $\pi_{i,t}^* \in \mathcal{A}$ and $U_i(x_1,\ldots,x_N,t) \in \mathcal{U}$, a.s.

ii) For all $\pi_{j,t}^* \in \mathcal{A}$, $j \neq i$, and each $\pi_{i,t} \in \mathcal{A}$, the process $U_i\left(X_{1,t}^{\pi_1^*},\ldots,X_{i,t}^{\pi_i},\ldots,X_{N,t}^{\pi_N^*},t\right)$ is a (local) supermartingale

iii) For all $\pi_{j,t}^* \in \mathcal{A}$, there exists $\pi_{i,t}^* \in \mathcal{A}$ such that $U_i\left(X_{1,t}^{\pi_1^*},\ldots,X_{i,t}^{\pi_i^*},\ldots,X_{N,t}^{\pi_N^*},t\right)$ is a (local) martingale

Construction of a forward Nash-equilibrium for linear wealth distortions

Let $i = 1, \ldots, N$, i^{th} player is endowed with measure $\nu_i \in \mathcal{V}(\nu)$

Forward Nash performance criterion

$$U_i(x_1, x_2, \ldots, x_N, t) = u_i\left(x_i - \theta_i \hat{x}, \int_0^t \lambda_s^2 \, \mathrm{d}s\right), \quad \hat{x} = \frac{1}{N}\sum_{j=1}^{N} x_i$$

with

$$u_{i,t} = \frac{1}{2}\frac{u_{i,x}^2}{u_{i,xx}} \qquad \text{with} \qquad \left(u'_{i,0}\right)^{-1}(x) = \int_{\mathbb{R}} \frac{x^{-y} - 1}{y} \nu_i(\mathrm{d}y)$$

Forward Nash control and controlled state processes

$$\hat{\theta} := \frac{1}{N}\sum_{j=1}^{N}\theta_i \quad \text{with} \quad \hat{\theta}\in[0,1); \quad A_t = \int_0^t \lambda_s^2\,\mathrm{d}s, \quad M_t = \int_0^t \lambda_s\,\mathrm{d}W_s$$

Nash policy processes

$$\pi_{i,t}^* = \frac{\lambda_t}{\sigma_t} h_{i,x}\left(h_i^{(-1)}(x_i - \theta_i\hat{x}, 0) + A_t + M_t, A_t\right)$$

$$+\frac{\lambda_t}{\sigma_t}\frac{\theta_i}{1-\hat{\theta}}\frac{1}{N}\sum_{j=1}^{N} h_{j,x}\left(h_j^{(-1)}(x_i - \theta_i\hat{x}, 0) + A_t + M_t, A_t\right)$$

Nash wealth processes

$$X_{i,t}^* = h_i\left(h_i^{(-1)}(x_i - \theta_i\hat{x}, 0) + A_t + M_t, A_t\right)$$

$$+\frac{\theta_i}{1-\hat{\theta}}\frac{1}{N}\sum_{j=1}^{N} h_j\left(h_j^{(-1)}(x_i - \theta_i\hat{x}, 0) + A_t + M_t, A_t\right)$$

If $\hat{\theta} = 0$, each player solves the forward problem in isolation

If $\hat{\theta} = 1$, there is no forward Nash equilibrium

Forward mean field games

Modeling the continuum of players

type vector $\zeta_i = (x_i, \theta_i, \nu_i)$ (independent of common noise $W = (W_1, \ldots, W_n)$)

type space $\mathcal{Z} := \mathbb{R} \times [0,1] \times \mathcal{V}$

type distribution $m_n(A) = \frac{1}{n} \sum \mathbf{1}_A(\zeta_i)$ for Borel sets $A \subset \mathcal{Z}$

weak limit $\int_{\mathcal{Z}} f \, \mathrm{d}m_n \longrightarrow \int_{\mathcal{Z}} f \, \mathrm{d}\mathbb{P}_0$, f bd, continuous

$\Downarrow$

type $\zeta = (x, \theta, \nu)$ in $\Omega_0 \times \mathcal{G} \times \mathbb{P}_0$

x **initial wealth,** θ **competition parameter,** ν **preferences**

$$h(z,0) = (u')^{-1}(z,0) = \int_{\mathbb{R}} \frac{e^{-zy} - 1}{y} \nu(\mathrm{d}y)$$

Defining the forward mean field game

Definition: A pair $(U(x,t),\pi^*)$, $t\geq 0$ is a forward MFG in $\left(\hat{\Omega},\hat{\mathbb{F}},\left(\hat{\mathcal{F}}_t\right),\hat{\mathbb{P}}\right)$ with $\hat{\Omega}=\Omega_0\times\Omega$, $\{\hat{\mathcal{F}}_t\}=\{\mathcal{G}\vee\mathcal{F}_t^W\}$, $\hat{\mathbb{P}}=\mathbb{P}_0\times\mathbb{P}$, if

- $U(x,t)\in\hat{\mathcal{F}}_t$, $x\in\mathbb{R}$, $t\geq 0$, and $U(x,t)$ is strictly increasing and concave
- for any $\pi\in\mathcal{A}_{[0,\infty)}^{MFG}$, the process

$$U\left(X_t^{\pi}-\theta E_{\hat{\mathbb{P}}}\left[X_t^{\pi^*}\,\middle|\,\mathcal{F}_t^W\right],t\right),\quad t\geq 0$$

  is a (local) supermartingale

- the process

$$U\left(X_t^{\pi^*}-\theta E_{\hat{\mathbb{P}}}\left[X_t^{\pi^*}\,\middle|\,\mathcal{F}_t^W\right],t\right),\quad t\geq 0$$

  is a (local) martingale

$$\mathcal{A}_{[0,\infty)}^{MFG}=\left\{\pi:\pi\in\hat{\mathcal{F}}_t,\ \text{self-financing and } E_{\hat{\mathbb{P}}}\left[\int_0^t\sigma_s^2\pi_s^2\,\mathrm{d}s\right]<\infty\right\}$$

Main result (Z. 2022)

Let $\bar{\theta} = E_{\mathbb{P}_0}(\theta)$ and $\bar{x} = E_{\mathbb{P}_0}(X_0)$; $\bar{\theta} \in (0,1)$

$$\alpha_t^{x-\theta\bar{x}} := \frac{\lambda_t}{\sigma_t} h_z\left(h^{(-1)}(x-\theta\bar{x},0) + \int_0^t \lambda_s \, \mathrm{d}W_s + \int_0^t \lambda_s^2 \, \mathrm{d}s, \int_0^t \lambda_s^2 \, \mathrm{d}s\right)$$

$$z_s^{*,x-\theta\bar{x}} := h\left(h^{(-1)}(x-\theta\bar{x},0) + \int_0^t \lambda_s \, \mathrm{d}W_s + \int_0^t \lambda_s^2 \, \mathrm{d}s, \int_0^t \lambda_s^2 \, \mathrm{d}s\right)$$

Forward MFG policy

$$\pi_t^{*,x} = \alpha_t^{x-\theta\bar{x}} + \frac{\theta}{1-\bar{\theta}} E_{\hat{\mathbb{P}}}\left[\alpha_t^{x-\theta\bar{x}} \middle| \mathcal{F}_t^W\right]$$

Forward MFG wealth

$$\mathrm{d}X_t^{*,x} = \mu_t \pi_t^{*,x} \, \mathrm{d}t + \sigma_t \pi_t^{*,x} \, \mathrm{d}W_t$$

$$X_t^{*,x} = z_t^{*,x-\theta\bar{x}} + \frac{\theta}{1-\bar{\theta}} E_{\hat{\mathbb{P}}}\left[z_t^{*,x-\theta\bar{x}} \middle| \mathcal{F}_t^W\right]$$

Going from the N-player game to the MFG

$$\pi_{i,t}^{*,x_i} = \alpha_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{1}{N}\sum_{j=1}^{N}\theta_i}\frac{1}{N}\sum_{j=1}^{N}\alpha_{i,t}^{*,x_i-\theta_i\hat{x}}$$

where

$$\alpha_{i,t}^{*,x_i-\theta_i\hat{x}} = \frac{\lambda_t}{\sigma_t}h_{i,x}\left(h_i^{(-1)}(x_i-\theta_i\hat{x},0)+\int_0^t\lambda_s\,\mathrm{d}W_s+\int_0^t\lambda_s^2\,\mathrm{d}s,\int_0^t\lambda_s^2\,\mathrm{d}s\right)$$

$$\Downarrow$$

$$\pi_t^{*,x} = \alpha_t^{x-\theta\bar{x}} + \frac{\theta}{1-\bar{\theta}}E_{\hat{\mathbb{P}}}\left[\alpha_t^{x-\theta\bar{x}}\middle|\mathcal{F}_t^W\right]$$

where

$$\alpha_t^{x-\theta\bar{x}} = \frac{\lambda_t}{\sigma_t}h_z\left(h^{(-1)}(x-\theta\bar{x},0)+\int_0^t\lambda_s\,\mathrm{d}W_s+\int_0^t\lambda_s^2\,\mathrm{d}s,\int_0^t\lambda_s^2\,\mathrm{d}s\right)$$

Examples

Forward SAHARA preferences (Musiela–Z. 2006)

$$\nu(\mathrm{d}y;\omega_0) = \frac{b}{2\kappa}(\delta_\kappa + \delta_{-\kappa});\ \kappa \in K(\omega_0),\ b \in B(\omega_0);\ h(x,0;\omega_0) = \frac{b}{\kappa}\cosh(\kappa x)$$

$$\Downarrow$$

Initial risk tolerance $r(x,0;\omega_0) = -\dfrac{u_x(x,0)}{u_{xx}(x,0)} = \sqrt{\kappa x^2 + b}, \quad x \in X(\omega_0)$

$$\Downarrow$$

$h(x,t) = \dfrac{b}{\kappa}\sinh(\kappa x)e^{-\frac{1}{2}\kappa^2 t}$ and $r(x,t) = \sqrt{\kappa x^2 + be^{-\kappa t}}$

Forward MFG policy

$$\pi_t^{*,x} = \frac{\lambda_t}{\sigma_t}\left(\sqrt{\kappa\left(z_t^{*,x-\theta\bar{x}}\right)^2 + be^{-\kappa\int_0^t \lambda_s^2\,\mathrm{d}s}}\right.$$

$$\left. + \frac{\theta}{1-\bar{\theta}} E_{\hat{\mathbb{P}}}\left[\sqrt{\kappa\left(z_t^{*,x-\theta\bar{x}}\right)^2 + be^{-\kappa\int_0^t \lambda_s^2\,\mathrm{d}s}} \,\middle|\, \mathcal{F}_t^W\right]\right)$$

Optimizing the probability of reaching a target

$$\nu(\mathrm{d}y) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2 t} \text{ or, more generally, } \nu(\mathrm{d}y;\omega_0) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\kappa(\omega_0)^2 t}$$

$$\Downarrow$$

$$u(z,t) = k_1 F\left(F^{(-1)}(z) - \sqrt{t+1}\right) + k_2; \quad F(z) = \int_0^t e^{\frac{1}{2}z^2}\,\mathrm{d}z \text{ and } f(z) = F'(z)$$

MFG forward criterion

$$U(x,t) = u(x-\theta\bar{x}, A_t); \qquad A_t = \int_0^t \lambda_s^2\,\mathrm{d}s, \quad M_t = \int_0^t \lambda_s\,\mathrm{d}W_s$$

MFG policy

$$\pi_t^* = \frac{\lambda_t}{\sigma_t}\left(\frac{1}{\sqrt{1+A_t}} f\left(\frac{F^{(-1)}(x-\theta\bar{x}) + M_t + A_t}{\sqrt{1+A_t}}\right)\right.$$

$$\left. + \frac{\theta}{1-\bar{\theta}}\frac{1}{\sqrt{1+A_t}} E_{\mathbb{P}}\left[f\left(\frac{F^{(-1)}(x-\theta\bar{x}) + M_t + A_t}{\sqrt{1+A_t}}\right) \middle| \mathcal{F}_t^W\right]\right)$$

Conclusions

- Formulated N-player games and MFG under forward performance criteria

- Solved MFG when controls appear in both the drift and the volatility

- Forward framework allows for richer criteria and more general markets
- Forward framework allows for a big class of utilities (beyond quadratic, exponential, power and logarithmic)