Dynamic contracting with many agents

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Abstract

We extend Merton's (1969) analysis of capital allocation and consumption-saving to the case in which asset management is delegated to many privately informed managers, whose compensation is linear in assets under management. With a continuum of managers and logarithmic utility, mean-field control techniques yield a simple solution: capital reallocation is linear in relative performance, the compensation rate is larger than under symmetric information and inversely proportional to the sensitivity to performance. Because incentive constraints limit risk-sharing, managers' continuation utilities are exposed to idiosyncratic risk and become increasingly unequal, while investment is lower than under symmetric information.

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1 Introduction

This paper revisits the canonical investment problem analyzed by Merton (1969), in which a risk averse agent dynamically chooses i) how much to consume or save and ii) how to allocate savings across risky assets. We revisit this problem to account for the fact that, in practice, the management of the investor's portfolio of risky assets is delegated to asset managers. Such delegation generates incentive problems, to the extent that the investor (the principal) and the asset managers (the agents) have different preferences and information sets. Thus, our goal is to study how optimal investment-consumption choices and portfolio allocations are altered by incentive constraints. We focus on the case in which the investor delegates the management of the assets to several asset managers, each with special expertise, and correspondingly private information, about a given asset class.

Our focus on investment delegation to several asset managers is in line with stylized facts. The principal in our model can be thought of as a sovereign wealth fund, a family office, a pension fund, or a foundation. The principal allocates his/her portfolio across several different classes of assets, e.g., private equity, private debt, real estate, or infrastructure. This allocation across assets coincides with an allocation across asset managers, investing in assets about which they have special expertise. For example, at the end of 2023, the Norwegian sovereign wealth fund had 742 billion kroner (approximately 67 billion US dollars), managed externally by 103 organisations.[1] Similarly, in 2025, the New Zealand Superannuation Fund used more than 40 different external asset managers.[2] Rose (2016) notes that "nearly all sovereign wealth funds make significant use of external managers, particularly for alternative asset classes that are difficult to access or demand labor intensive investment strategies, such as private equity, venture capital, and hedge fund strategies." This quote suggests that delegation is motivated by the need to rely on the skills of the managers, skills which the investor does not master. In the Invesco Global Sovereign Asset Management Study, respondents stated that they often relied on external asset managers, and one of the respondents declared:[3] "When... we have been forced to operate the business ... that has historically not worked well, so we are moving away from that model and trying to leave all management to general partners". However, when delegation is motivated by the superior understanding of assets by agents, this generates an information asymmetry between the investor and the asset managers. This information asymmetry implies that the optimal allocation

[1] See https://www.nbim.no/en/the-fund/how-we-invest/external-mandates. The Norwegian Ministry of Finance decides the allocation of the portfolio across broadly defined asset classes, and then the sovereign fund decides how to allocate funds within these classes to different asset managers.

[2] See https://nzsuperfund.nz/how-we-invest/investment-managers/our-managers/

[3] See https://ioandc.com/sovereign-wealth-funds-tap-external-managers-for-private-markets-payday/

of assets across managers is subject to incentive compatibility constraints.

We model the interaction between an investor and N managers investing in different asset classes, such as private equity, private debt, real estate, or infrastructure. Each manager invests in a specific asset class over which he/she has expertise. The unit return is independent of the quantity invested, i.e., there are constant returns to scale. For simplicity, all agents are equally skilled, i.e., they have the same expected return per unit. They, however, are subject to different random shocks: each agent's return is hit by an agent-specific idiosyncratic shock, as well as a common aggregate shock. The aggregate shock and the allocation of capital across agents are publicly observable, and capital allocation is decided by the principal.

To model information asymmetry between the investor and the asset managers, we assume that agents privately observe their idiosyncratic shock, and correspondingly their return, which they can secretly divert and consume, as in Bolton and Scharfstein (1990).[4] We study the optimal mechanism designed by the investor (the principal) to allocate capital and compensation to the asset managers (the agents).[5] The incentive compatibility constraint is that agents prefer to truthfully report their privately observed returns. To analyze incentive constraints, we rely on martingale techniques, as Schättler and Sung (1993) and Sannikov (2008). Incentive compatibility implies that agents' utilities must be sensitive to idiosyncratic shocks. Intuitively, after good performance agents are rewarded, while after bad performance they are punished. Differences in performance, therefore, generate endogenous heterogeneity among agents.

With a finite number of agents the optimal mechanism is complex due to the interactions between risk sharing between asset classes and incentive constraints for each manager. With a continuum of agents, the idiosyncratic risks of the different asset classes is fully diversified at the aggregate level. As explained below this leads to a simpler optimal mechanism.

To solve for the optimal mechanism, however, we have to solve a challenging mathematical problem. The distribution of the continuation utilities of the different agents is a state variable of the control problem of the principal. With a finite number of agents, this state variable is simply a vector. With a continuum of agents, the distribution of utilities across agents is an infinite dimensional object. Moreover, with aggregate risk it evolves stochastically. To analyze the optimal mechanism in that context we need to perform Itô calculus with infinite dimensional variables and to write a Hamilton Jacobi Bellman equation in which the value function is a functional. To do so we

[4]Another possible specification is that agents privately observe the effort they exert to improve returns, as in Holmström and Tirole (1997). In DeMarzo and Fishman (2007a) and DeMarzo and Sannikov (2006) privately observed returns and privately observed effort lead to similar results.

[5]The principal signs one contract with each agent, the collection of these contracts is the mechanism.

rely on mean-field techniques.[6]

For general mechanisms, the combination of our assumptions of a continuum and constant returns to scale creates a non-existence problem. However, existence is restored by restricting the analysis to linear mechanisms, i.e., mechanisms in which consumption is linear in assets under management. This class of mechanisms is in line with the linear compensation schemes commonly observed in practice and considered in most theoretical analyses, see, e.g., Berk and Green (2004). Within this class of mechanisms, the principal chooses the linear compensation rate and the allocation of capital, and this determines the exposure of the managers' continuation utilities to their return shocks. The restriction to compensation schemes that are linear in assets under management is not without loss of generality but, in the case of logarithmic utilities, it enables us to obtain quasi analytical solutions yielding the following economic insights:

- Although on the equilibrium path performance only reflects luck, incentive compatibility requires that agents with better (resp. worse) idiosyncratic performance be rewarded (resp. punished). This generates exposure to idiosyncratic risk, reducing welfare relative to the first-best, in which idiosyncratic risk is fully mutualized, as in Borch (1962).

- The link between performance and rewards goes through capital allocation. Agents with good performance are allocated more capital than agents with bad performance. This implies that capital allocation becomes more and more heterogeneous with time, with successful managers being allocated an increasingly larger fraction of capital.

- Capital reallocation among managers is linear in their relative performance. Since individual performances follow geometric Brownian motions with constant parameters, assets under management also follow a geometric Brownian motion with constant parameters.[7]

- Since incentive compatibility requires that managers be exposed to the idiosyncratic component of their performance in contrast with the first

[6]In physics, mean-field theory studies the behavior of models where a large number of particles interact randomly by studying a simpler model where each particle interacts with an average distribution (a mean-field) of particles. The mathematical theory of mean-fields is presented in Lasry and Lions (2007), Cardaliaguet (2012), and Carmona and Delarue (2018). As discussed in more detail below, we take a mean-field *control* approach, which is the appropriate tool to study mechanism design and contracting, and is mathematically different from mean-field *games*, appropriate to study equilibrium.

[7]In a single-agent model Pegoraro (2023) also finds that the principal delegates more capital to the manager after good performance. This reflects learning about the manager's productivity. In contrast in our model, all managers are equally skilled, performance only reflects luck, and there is no learning.

best, to satisfy the agents' initial participation constraint fees must be higher in the second best than in the first best. Moreover, fees are inversely proportional to the sensitivity of continuation utility to performance.

- Since incentives compatibility precludes perfect sharing of idiosyncratic risk in contrast with the first best, holding capital comes at the cost of greater risk exposure in the second best than in the first best. This lowers the optimal second best investment rate below its first part counterpart.

The simple and intuitive results we obtain with a continuum of agents are in stark contrast with those obtaining with a single agent.[8] With a single agent, there is no quasi closed-form solution and any characterization of the optimal contract is numerical. In contrast with the continuum of agents' case the sensitivity of the agent's utility to performance fluctuates randomly. Intuitively, the reason for the difference between the one-agent case and the many-agents case is the following: With a single agent (or a small number of agents), the resources needed to provide individual incentives are significant relative to the total amount of resources available for investment and consumption. This generates a direct interaction between the individual incentive problem of an agent and the overall investment-consumption choice of the principal. In contrast, with a continuum of agents, agents' idiosyncratic shocks, and correspondingly, the resources needed to incentivize an individual agent, are negligible relative to aggregate resources. So, the individual incentive problems of the agents are separable from the investment-consumption problem of the principal. It is this separability that makes the problem tractable.

Our theoretical findings are consistent with stylized facts about asset management. Our model implies that asset managers with better performance should attract more capital, which is in line with the empirical results of Chevalier and Ellison (1997) and Sirri and Tufano (1998). This takes place, in our model, in spite of the fact that all managers are equally skilled, so that realized performance does not reflect skills and does not predict future performance. The latter is consistent with the empirical finding that fund performance is hard to predict out of sample (see, e.g., Jones and Mo (2021)). Our theoretical model also implies that the distribution of capital and fees across managers should be skewed to the right, which is in line with empirical findings (see Bai et al, 2024). Collecting panel data on the allocation of wealth by large investors (e.g., Sovereign Wealth Funds) to different asset managers and asset classes, would enable one to conduct further empirical tests of our

[8]The characterization of the dynamic optimal contract between a risk averse principal and a single agent is very difficult and has not been offered by previous literature (e.g., in Sannikov (2008) the principal is risk-neutral). But, of course, investor's risk aversion is a necessary ingredient of the Merton (1969) problem we revisit, in which risk-return tradeoffs play a key role.

theory. For example, one could test the implications of our model for the joint dynamics of the cross-section of agents' performances and capital allocation.

The rest of the paper has the following structure. Section 2 discusses the relationship between our paper and the literature. Section 3 presents the model. Section 4 considers the symmetric information benchmark, characterizing the first-best optimal allocations. Section 5 turns to the case of asymmetric information with a finite number of agents, characterizing, in particular, the incentive compatibility condition for a general number of agents, and the optimal mechanism with a single agent. Section 6 considers the case of a continuum of agents, characterizing, in particular, the functional Hamilton Jacobi Bellman equation. Section 7 presents an explicit solution of the optimal control problem for logarithmic utilities. Section 8 concludes. Appendix A.1 provides proofs and propositions for the case of symmetric information not provided in the main text. Appendix A.2 details the numerical analysis of the one-agent case, and Appendix A.3 provides the formal statements of admissible mechanisms and the proofs for the case of a continuum of agents. Appendix B, available online, offers an introduction to the mean-field tools that underlie our analysis. The second online Appendix C presents extensions of our basic model to the case of a risk-neutral principal and to the case in which there is a risk free asset.

2 Literature

2.1 Continuous time finance and economics

Our analysis is at the cross of three strands of the continuous-time literature in finance and economics:

First, our analysis of dynamic portfolio allocations and consumption-savings choices is in the line of Merton (1969, 1973, 1987). The contribution of our analysis relative to these papers is to allow for asymmetric information between investors and asset managers, implying that capital allocation is not only driven by risk and return considerations but also by incentive considerations.

Second, to factor in these incentive considerations, we take an optimal dynamic contracting approach, in line with the seminal contributions of DeMarzo and Fishman (2007a, 2007b) and Sannikov (2008).[9] Two key differences between our paper and that literature are that we study the optimal

[9]See also DeMarzo and Sannikov(2006), Biais, Mariotti, Plantin, and Rochet (2007), Biais, Mariotti, Rochet, Villeneuve (2010), DeMarzo, Fishman, He and Wang (2012), Feng and Westerfield (2021), Yang (2020), Di Tella and Sannikov (2021), Dai, Wang, and Yang (2024), and Gryglewicz and Mayer (2023). In the present paper, as in Biais, Mariotti, Rochet, and Villeneuve (2010) and DeMarzo, Fishman, He, and Wang (2012), the scale of operations is determined by the optimal contract and is useful to provide incentives. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is

contract between i) a *risk-averse* principal and ii) *several* agents. Considering a risk-averse principal is necessary to study a Merton problem in which the principal is an investor who evaluates risk-return tradeoffs. Considering several agents is appropriate to fit the stylized fact that the majority of large investors, such as sovereign wealth funds, family offices, or pension funds, rely on several asset managers.

Third, our analysis is related to the mean-field literature, see, e.g., Lasry and Lions (2007), Cardaliaguet (2012), Carmona and Delarue (2018), Elie, Mastrolia, and Possamaï (2019), and Carmona (2020). We study a mean-field control problem which is related to, but different from, mean-field games. Achdou et al. (2022) use mean-field games in a macroeconomic model to study market equilibrium. In contrast, we use mean-field control in a contract-theory model to study incentive-constrained optimal allocations. The methodological difference between mean-field games and mean-field control is illustrated by the difference between the analysis of Achdou et al (2022) and our analysis:

- In the mean-field game approach of Achdou et al. (2022), each agent solves a Hamilton Jacobi Bellman equation, and the equilibrium distribution of wealth among agents is determined by a Fokker-Planck equation.

- In our mean-field control approach, it is the principal who solves a Hamilton Jacobi Bellman equation, and the distribution of agents' utilities is a state variable of that optimization problem.

Elie, Mastrolia, and Possamaï (2019), and Carmona (2020) study mean field control problems, but the models and issues they analyze differ from those we analyze. Section 6 of Carmona (2020) bears on moral hazard. It focuses on a discrete state, finite horizon model, and shows that the analysis of the model reduces to the solution of a system of backward-forward ODEs which can be solved numerically. The economic application of the analysis in Carmona (2020) is the optimal policy mitigating the spread of a contagious disease. Elie, Mastrolia, and Possamaï (2019) analyze a finite horizon moral hazard problem with infinitely many agents via a functional Hamilton Jacobi Bellman equation. Their economic application bears on the special case in which the principal and the agents consume only at the terminal point in time. Relative to that literature, our mathematical contribution is to offer a solution to the functional Hamilton Jacobi Bellman equation in an infinite horizon model in which the principal and the agents consume at all points in time.[10] Our economic contribution is to use this framework to shed light on an incentive constrained capital allocation and risk-sharing.

output.

[10]To handle functionals defined on a set of probability measures with a finite exponential moment, we use first and second variations along mixtures of measures (linear functional derivatives, Carmona and Delarue (2018), Vol. 1, Chapter 5). However, the Itô formulas established on the Wasserstein space do not apply to our value func-

2.2 Asset management

In contrast with our asymmetric information model without learning, Berk and Green (2004) analyze a model in which the investor and the asset managers have the same information and conduct learning about the managers' skills. Thus, in Berk and Green (2004) portfolio allocation is first-best efficient, unlike in our model. Moreover, Berk and Green (2004) consider linear utilities. This contrasts with our risk-aversion assumption, which, in conjunction with our information asymmetry assumption, generates inefficiencies in portfolio allocation and investment-consumption choices.

In the main analysis of Binsbergen, Brandt and Koijen (2008) the investor and the asset manager have the same information, so any inefficiency in capital allocation and diversification is due to suboptimal contracts. Binsbergen, Brandt and Koijen (2008) then analyze the case in which the investor does not know the risk aversion of the asset managers, but they don't study the incentive-constrained optimal contract between the investor and the agents.

He and Xiong (2013) and Ou-Yang (2003) analyze optimal contracts. In their models, however, there is only one agent, and capital is allocated once and for all at the beginning of the contract. So, our analysis of dynamic reallocation of capital among agents brings a new contribution relative to He and Xiong (2013) and Ou-Yang (2003).

2.3 Optimal contracts with several agents

As in Holmström (1982), the principal in our model interacts with several privately informed agents. But the focus of Holmström (1982) is very different from ours. In Holmström (1982) the actions of the agents interact in the determination of aggregate output. This can generate free-riding, which is a major issue in Holmström (1982). In contrast in our paper, agents' individual returns are independent from one another (conditional on the aggregate shock), so there is no free-riding, which simplifies the analysis. On the other hand, our paper studies the dynamic allocation of capital across agents, an issue that is absent from Holmström (1982).

3 Model

We extend Merton(1969)'s analysis of consumption, investment, and portfolio allocation to a principal-agent context in which the investor (the principal) delegates the management of the portfolio of assets to several asset managers (the agents).

3.1 Preferences and Technology

Time is continuous: $t \in (0, \infty)$. The agents indexed by $i = 1, ...N$ and the principal are infinitely lived with discount rate ρ and utility $\rho u(c)$, where

tion, which is not continuous for the Wasserstein distance. In online Appendix B, we directly prove the Itô formula we need (Appendix B.2).

$u(.)$ is increasing and concave.[11] To illustrate our results, we will take the specification $u(c) = \log c$.

Each of the N agents has mass $1/N$, so that the total mass of the agents is equal to 1. We will later take the limit when N goes to infinity to consider a continuum of agents each with infinitesimal mass. There is a single good which can be consumed or invested in N stochastic constant returns to scale technologies.

Concavity of the principal's utility not only generates risk aversion, it also impacts intertemporal choices: As explained in the online appendix, a principal with linear utility could find it optimal to backload consumption, which could give rise to infinite value. In contrast, with concave utility the principal prefers to smooth out consumption over time.

The principal does not have the expertise to operate the production technologies, but the agents do. More precisely, each agent has the expertise to operate one of the N technologies. The capital invested by agent i is $\frac{k_t^i}{N}$ and his consumption flow is $\frac{c_t^i}{N}$. Agent i's time t investment, $\frac{k_t^i}{N}$, generates instantaneous output:

$$dY_t^i = \frac{k_t^i}{N}[\mu dt + \sigma dZ_t^i + \sigma_A dZ_t^A], \tag{1}$$

where μ is the expected rate of return of the technology and $(Z_t^A, Z_t^i)_{i=1,...N}$, are independent Brownian motions. (dZ_t^A) is an aggregate productivity shock, to which all projects are equally exposed, while $(dZ_t^i)_{i=1,...N}$ are idiosyncratic, project-specific, productivity shocks. $(\mathcal{F}_t)_{t\geq 0}$ denotes the augmented filtration generated by the $N+1$-dimensional Brownian motion $(Z_t^A, Z_t^1, \ldots, Z_t^N)_{t\geq 0}$. All processes introduced in this section are assumed to be progressively measurable with respect to $(\mathcal{F}_t)_t$.

3.2 Aggregate capital and output

The total amount of capital in the economy at time t is

$$K_t := \frac{1}{N}\sum_{i=1}^{N} k_t^i. \tag{2}$$

At time t, the consumption flow of agent i is $\frac{c_t^i}{N}$, the consumption of the principal is c_t^P, and the law of motion of aggregate capital is

$$dK_t = \sum_{i=1}^{N} dY_t^i - \left(\frac{1}{N}\sum_{i=1}^{N} c_t^i + c_t^P\right) dt. \tag{3}$$

[11] Multiplying the current utility by ρ is a useful convention that generates simpler formulas. For example, the intertemporal utility of a constant consumption flow c is just $\int_0^\infty \rho e^{-\rho t} u(c) dt = u(c)$.

Summing over the individual outputs given in (1) and using (2), the aggregate output is given by

$$dY_t = \sum_{i=1}^{N} dY_t^i = K_t(\mu dt + \sigma_A dZ_t^A) + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^i dZ_t^i. \tag{4}$$

Substituting (4) into (3), the dynamics of aggregate capital rewrites

$$dK_t = \left(\mu K_t - \frac{1}{N}\sum_{i=1}^{N} c_t^i - c_t^P\right) dt + \sigma_A K_t dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^i dZ_t^i. \tag{5}$$

Equation (5) is a resource constraint stating that net aggregate investment is equal to total output minus total consumption.[12]

3.3 Pareto frontier

A consumption path for an agent is a progressively measurable process (c_t) such that

$$\mathbb{E}\int_0^\infty \rho e^{-\rho s}|u(c_s)|ds < +\infty. \tag{6}$$

For a given consumption path (c_t^i), the expected continuation utility of agent i at time t is

$$\omega_t^i := \mathbb{E}_t \int_t^\infty \rho e^{-\rho(s-t)} u(c_s^i) ds. \tag{7}$$

We seek to characterize the Pareto frontier of the economy by computing the value function V of the principal, defined as the maximum expected utility she can obtain with an initial amount of capital K when agent $i \in (1, ..., N)$ gets the initial expected utility ω_0^i. Let us denote by $\mathbb{W}$, the vector of initial expected utilities $(\omega^1, ..., \omega^N)$. $\mathbb{W}$ are the outside utilities that the agents could obtain initially if they refused to contract with the principal. They can be interpreted as reflecting what the agents could obtain from contracting with other principals competing with the principal analyzed in our model.

The value function of the principal is obtained by finding capital and consumption paths (k_t^i, c_t^i, c_t^P) that maximize the expected utility of the principal.

$$\mathbb{E}\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt, \tag{8}$$

subject to the relevant constraints. With symmetric information, these constraints are the capital allocation constraint (2), the resource constraint (5), and the initial conditions: $K_0 = K$ and $\omega_0^i = \omega^i$, for $i = 1, ..., N$. Under information asymmetry, there will also be incentive compatibility constraints.

[12] For brevity, we do not explicitly include depreciation. μ can be understood as capital productivity net of depreciation.

We assume that the agents and the principal commit to the contract signed on the date 0.[13] We first consider the case of symmetric information, in which idiosyncratic shocks and thus individual outputs are publicly observable. This case will then serve as a reference point for the case in which agents privately observe their idiosyncratic shocks.

3.4 Promise keeping

Denote by M_t^i the expectation of the lifetime utility of agent i conditional on information at time t :

$$M_t^i \equiv \mathbb{E}_t \left[\int_0^\infty \rho e^{-\rho s} u(c_s^i) ds \right] = \int_0^t \rho e^{-\rho s} u(c_s^i) ds + e^{-\rho t} \omega_t^i.$$

Consequently, the dynamics of ω_t^i is

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i) \right] dt + e^{\rho t} dM_t^i.$$

Since M_t^i is the conditional expectation of an integrable random variable, it is a martingale. Therefore, by the Martingale Representation Theorem, dM_t^i can be written as a linear combination of Brownian shocks at time t, $(dZ_t^j)_{j=1,...,N}$ and (dZ_t^A). Thus, the dynamics of i's continuation utility can be written as

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i) \right] dt + \frac{\sigma}{N} \sum_{j=1}^N \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A, \tag{9}$$

where β_t^{ij} and $\beta_t^{A,i}$ are adapted to $(\mathcal{F}_t)$. $\beta_t^{A,i}$ can be interpreted as the share of the volatility of the aggregate shock that is borne by agent i, while β_t^{ij} is the exposure of agent i to the idiosyncratic shock of agent j. They are controls chosen by the principal. In line with the contract theory literature, hereafter we refer to equation (9) as the "promise keeping condition ".

4 Optimal allocations under symmetric information

The focus of this paper is on the asymmetric information case. However, the symmetric information case offers a useful benchmark.

4.1 The control problem of the principal

The state variables of the principal's control problem are the capital stock K_t and the vector $\mathbb{W}_t \equiv (\omega_t^1, ..., \omega_t^N)$ of promised utilities. The controls k_t^i,

[13] Hence, agents cannot walk away from the contract after time 0 to take an outside option. While this assumption may be unrealistic, it is needed to simplify the analysis, which is already quite complex. Note however that our commitment assumption does not necessarily imply that agents' continuation value becomes arbitrarily negative. As explained below, if capital productivity is high enough all agents' utilities go to infinity.

c_t^i, c_t^P, β_t^{ij}, and $\beta_t^{A,i}$ will only depend on K_t and $\mathbb{W}_t$, as well as individual characteristics.[14]

The value function of the principal is

$$V(K,\mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt\right], \tag{10}$$

subject to the capital allocation constraint (2), the resource constraint (5), the promise keeping condition (9), and the initial conditions: $K_0 = K$ and $\mathbb{W}_0 = \mathbb{W}$. The associated Hamilton-Jacobi-Bellman equation is

$$\begin{aligned}\rho V(K,\mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} & \rho u(c^P) + \sum_{i=1}^N \left[V_{\omega^i}\rho\left(\omega^i - u(c^i)\right)\right] \\ & + V_K\left(\mu K - c^P - \frac{1}{N}\sum_{i=1}^N c^i\right) \\ + \sum_{i=1}^N \left[V_{\omega^i K}\left(\beta^{A,i}\sigma_A^2 K + \sigma^2 \sum_{j=1}^N \frac{k^j \beta^{ij}}{N^2}\right)\right] & + \frac{1}{2}V_{KK}\left[\sigma_A^2 K^2 + \sigma^2 \sum_{i=1}^N \left(\frac{k^i}{N}\right)^2\right] \\ & + \frac{1}{2}\sum_{i,j} V_{\omega^i\omega^j}\left(\sigma_A^2 \beta^{A,i}\beta^{A,j} + \sigma^2 \sum_{\ell=1}^N \frac{\beta^{i\ell}\beta^{j\ell}}{N^2}\right)\end{aligned} \tag{11}$$

where the sup is subject to the capital allocation constraint (2). The shape of the Hamilton-Jacobi-Bellman equation (11) implies that in the first-best, consumption allocation and capital allocation are separable, since the consumption controls c^i and c^P appear only in the first line of (11), while the capital allocation k appears only in the second line.

We call a mechanism $X = (k, \beta, \beta^A, c, c^P)$ admissible if $K_t^X > 0$ and $k_t^i \geq 0$ for all t, if the consumption paths of the principal and of the agents satisfy the integrability condition (6), and if, for every agent i, the stochastic integral in $M_t 1^i$ in Section 3.4) is a martingale and $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}[\omega_T^i] = 0$, so that ω_t^i is the continuation utility (7) of agent i.

The verification theorem in stochastic control that we recall in the next proposition (see, e.g., Yong and Zhou (1999) chapter 4) will allow us to identify the optimal mechanism:

Proposition 1. *Let V be a twice continuously differentiable solution of the HJB equation (11) on an open set that contains the states $(K_t^X, \mathbb{W}_t^X)$ of all admissible mechanisms X. Assume that, for every admissible mechanism X, the stochastic integrals in Itô's formula for $e^{-\rho t}V(K_t^X, \mathbb{W}_t^X)$ are martingales and the transversality condition*

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}[V(K_t^X, \mathbb{W}_t^X)] = 0$$

[14] Note that these processes are controlled, and when necessary, they will be denoted $(K_t^X, \mathbb{W}_t^X)$ along a control $X = (k, \beta, \beta^A, c, c^P)$.

holds. Then $V(K, \mathbb{W})$ is at least as great as the principal's expected utility (8) under any admissible mechanism. If, moreover, the supremum in (11) is attained at every date by the controls of an admissible mechanism X^*, then V is the value function of the principal's problem (10) and X^* is optimal.

4.2 Explicit solution when utility is logarithmic

Merton (1969) studied a simpler form of our model in which the investor did not need agents and directly managed the portfolio of assets. In that case, when $u(c) = \log c$, the solution always exists and the optimal allocation is such that:

- capital is equally allocated across assets,[15]
- the consumption flow is proportional to the capital stock,
- the capital stock follows a geometric Brownian motion,
- the value function of the principal satisfies $V(K) = \log mK$ for some constant m.

We now show that a similar result obtains when the principal must delegate asset management to $N \geq 1$ agents under symmetric information.

However, a solution to this problem exists only if the capital stock is large enough to provide the continuation utilities promised to the agents at date 0. More specifically, we need

$$K_0 > \frac{A_0}{m_N}, \tag{12}$$

where A_t is the cross sectional average of the inverse utilities of the agents:

$$A_t := \frac{1}{N} \sum_{i=1}^{N} \exp(\omega_t^i), \tag{13}$$

and the constant m_N is given by

$$m_N = \rho \exp(\frac{1}{\rho}(\mu - \rho - \frac{1}{2}(\sigma_A^2 + \frac{\sigma^2}{N})). \tag{14}$$

The next proposition characterizes the first-best allocations between the principal and the agents:

Proposition 2. *When u is logarithmic and $N \geq 1$, the optimal mechanism exists if and only if (12) holds. In that case, the value function of the principal is*

$$V(K, \mathbb{W}) = \log K + \log(m_N - \frac{A}{K}), \tag{15}$$

and the optimal mechanism has the following features for all t:

[15] This is because, in our model, all assets have the same expected return and volatility.

- *Capital is equally allocated across agents:*

$$k_t^i \equiv K_t, i = 1, ..., N. \tag{16}$$

- *The consumption flows of the principal and the agents are proportional to capital:*

$$c_t^P = \gamma K_t, c_t^i = \gamma^i K_t, i = 1, ..., N. \tag{17}$$

- *The aggregate consumption rate is equal to the discount rate*

$$\gamma^P + \frac{1}{N}\sum_{i=1}^{N} \gamma^i = \rho \tag{18}$$

- *The inverse utility of each agent is proportional to aggregate capital:*

$$\exp \omega_t^i = \frac{K_t}{K_0} \exp \omega_0^i, \tag{19}$$

and therefore the average inverse utility of the agents, A_t, is also proportional to aggregate capital:

$$A_t = \frac{K_t}{K_0} A_0. \tag{20}$$

- *Aggregate capital and individual capital follow a geometric Brownian motion:*

$$\frac{dK_t}{K_t} = \frac{dk_t^i}{k_t^i} = \left(\mu - \gamma^P - \frac{1}{N}\sum_{i=1}^{N} \gamma^i\right) dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} dZ_t^i. \tag{21}$$

- *The sensitivity of agent i's continuation utility to the Brownian shocks is given by*

$$\beta^{A,i} \equiv \beta^{ij} \equiv 1, \quad i, j = 1, ..., N. \tag{22}$$

The proof of Proposition 2 does not rely on Proposition 1. It takes advantage of the specific features of the log-utility case to offer a direct upper bound, valid for every admissible mechanism (see Appendix A.1).

Equations (16) and (17) show that the unconstrained optimal mechanism obtained with symmetric information falls within the class of linear compensation mechanism, as the compensation of an agent, which is equal to the consumption of this agent, is equal to a coefficient multiplied by the capital allocated to this agent, i.e., this agent's assets under management. So, with finitely many agents, symmetric information and log-utility, restricting attention to linear compensation mechanisms is without loss of generality. With asymmetric information this will no longer be the case.

Equation (20) states that the ratio of A_t to aggregate capital is constant and Equation (15) shows that the value function depends on the vector of

continuation utilities $\mathbb{W}_t$ only through this ratio. In the asymmetric information case similar properties will obtain for linear compensation mechanisms, which will faclitate the analysis of these mechanisms.

Proposition 2 illustrates that with log utility and symmetric information, the capital allocation problem and the consumption allocation problem can be solved separately.

- The sole purpose of capital allocation is to maximize diversification. Thus, as stated in Equation (16), capital is allocated equally between projects run by the different agents.

- As can be seen in (16) and (17), an agent's consumption and capital only depend on aggregate capital. Hence, an agent's individual performance affects him/her only via its impact on aggregate capital. Thus, as stated in Equation (22), idiosyncratic risks and aggregate risk are equally shared among all participants. However, that the individual performance of an agent equally affects this agent and the other agents is at odds with stylized facts. As shown in the next section, with asymmetric information we will obtain the more plausible result that an agent's performance affects this agent more than it affects the others.

5 Optimal allocations under asymmetric information

We now turn to the case in which the managers privately observe their idiosyncratic shock (dZ_t^i) and can secretly divert and consume some of the corresponding output, as in Bolton and Scharfstein (1990) and DeMarzo and Sannikov (2006).[16] The incentive compatibility condition requires that agents truthfully report their shocks to the investor. We consider linear compensation mechanisms mapping the history of aggregate and idiosyncratic shocks into allocations, under the resource and incentive compatibility constraint.

Heuristically, the sequence of events at time t is the following:

- Each agent i operates the technology, with capital k_t^i.

- Shocks dZ_t^i and dZ_t^A realize. Each agent i reports $d\hat{Z}_t^i$ to the planner, transferring the corresponding output $d\hat{Y}_t^i$.

- As a function of the history of shocks, the planner allocates to the agent i, his/her period consumption flow c_t^i, capital for the next period k_{t+dt}^i, and promised continuation utility ω_{t+dt}^i. The agent consumes what he/she eceived from the principal as well as any resources he/she absconded with.

[16] In contrast to idiosyncratic shocks, aggregate shocks and individual capital are still publicly observable.

5.1 Incentive compatibility and the Hamilton Jacobi Bellman equation

By the promise keeping condition (9), when truthfully revealing dZ_t^i and anticipating that other agents report truthfully, agent i gets

$$\rho u(c_t^i)dt + \frac{\sigma}{N}\sum_{j=1}^{N}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{23}$$

Similarly, when under-reporting: $d\hat{Z}_t^i = dZ_t^i - \delta dt$, agent i gets

$$\rho u(c_t^i + \sigma\delta k_t^i)dt + \frac{\sigma}{N}\beta_t^{ii}(dZ_t^i - \delta dt) + \frac{\sigma}{N}\sum_{j\neq i}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{24}$$

Incentive compatibility requires (23) to be larger than (24). Because u is concave, this inequality holds for all $\delta \geq 0$ under the condition stated in the following proposition:[17]

Proposition 3. *The incentive compatibility condition holds if and only if*

$$\beta_t^{ii} \geq \rho N k_t^i u'(c_t^i). \tag{25}$$

Since $k_t^i u'(c_t^i) > 0$, (25) implies that the dynamics of ω_t^i in (9) must be affected by the idiosyncratic shock of agent i. Substituting the first best allocation obtained for logarithmic utility, given in (16), (17), and (22), into (25), implies $\gamma^i \geq N\rho$. This, however, is inconsistent with (18) and $\gamma^P > 0$. So, at least for logarithmic utility functions, the first best allocation is not incentive compatible. This reflects that in the first best $\beta_t^{ii} = 1$, i.e., the idiosyncratic risk of each agent is fully mutualized by all the agents. This contradicts the incentive compatibility constraint (25), which requires that agents be sufficiently exposed to their own shock.

Moreover, the incentive compatibility condition (25) gives rise to a trade-off between risk sharing and investment. Exposing risk-averse agents to idiosyncratic shocks reduces their utility. To reduce that exposure, one needs to reduce β_t^{ii}. However, due to (25), reducing β_t^{ii} requires increasing c_t^i and/or reducing k_t^i. As shown below, the second best will thus differ from the first best by tilting the consumption trade-off towards consumption, leading to less investment than in the first best.

With asymmetric information, the Hamilton Jacobi Bellman equation is still given by (11) but the maximization problem is subject to the incentive

[17] In our cash-diversion model à la Bolton Scharfstein (1990), agents privately observe their output after it is realized and can secretly divert it. A moral-hazard model à la Holmström and Tirole (1997) leads to the same incentive compatibility condition (25) under the assumption that shirking reduces the drift of the output process by $\sigma\delta$ per unit of capital and gives private benefit $\sigma\delta k_t^i$ in units of consumption. DeMarzo and Sannikov (2006) demonstrate the similarity between the two forms of information asymmetry in a one-agent model with risk neutrality.

compatibility condition (25), in addition to the capital allocation constraint (2). Condition (25) implies that, in contrast to the first best, capital allocation and consumption allocation decisions are no longer separable. This is because, while in (11) the terms involving capital k^i and consumption c^i are added to one another, in (25) capital k^i and consumption c^i multiply one another. Non separability of capital and consumption decisions makes the optimal control problem much less tractable. To illustrate this difficulty, we now study the simplest version of this problem: with a single agent ($N = 1$) and logarithmic utilities. It turns out that this problem can only be characterized partially, but cannot be solved explicitly.

5.2 The control problem with a single agent

When $N = 1$, the Hamilton Jacobi Bellman equation (11) becomes:

$$\rho V(K,\omega) = \sup_{\beta,\beta^A,c,c^P} \rho u(c^P) + \rho V_\omega \left[\omega - u(c)\right] + V_K \left(\mu K - c^P - c\right)$$

$$+V_{\omega K}\left(\beta^A\sigma_A^2 + \beta\sigma^2\right)K + \frac{1}{2}V_{KK}\left[\sigma^2 + \sigma_A^2\right]K^2 + \frac{1}{2}V_{\omega\omega}\left(\sigma_A^2(\beta^A)^2 + \sigma^2\beta^2\right) \tag{26}$$

where the sup is taken under the incentive compatibility constraint

$$\beta \geq \rho K u'(c). \tag{27}$$

With logarithmic utilities, optimal mechanisms are homogeneous: if the capital stock is multiplied by $\lambda > 0$, the optimal consumption flows are also multiplied by λ, and the utilities of both the agent and the principal are increased by an amount $\log\lambda$. That is

$$V(\lambda K, \omega + \log\lambda) = V(K,\omega) + \log\lambda.$$

Taking $\lambda = 1/K$, this yields

$$V(1, \omega - \log K) = V(K,\omega) - \log K. \tag{28}$$

Denoting

$$a \equiv \omega - \log K \tag{29}$$

and $v(a) \equiv V(1,a)$, (28) becomes

$$V(K,\omega) = \log K + v(a). \tag{30}$$

This homogeneity can be exploited by redefining the control variable

$$\gamma = \frac{c^P}{K}.$$

Using the above and evaluating the partial derivatives in (26) yields the next proposition.

Proposition 4. *Suppose there is only one agent, the principal and the agent have logarithmic utility, and $V(K,\omega) = \log K + v(a)$ with v twice differentiable. If the right-hand side of (26) is finite at (K,ω), then $v'(a) < 1$ and $v''(a) \leq 0$. If moreover $v''(a) < 0$, the supremum in (26) is attained, aggregate risk is equally shared between the principal and the agent ($\beta^A = 1$), the incentive compatibility condition (27) binds, and (26) reduces to*

$$\rho v(a) = \sup_{\gamma,\beta} \rho \log \gamma + \rho v'(a)(a - \log \frac{\rho}{\beta})$$

$$+(1 - v'(a))(\mu - \gamma - \frac{\rho}{\beta} - \frac{\sigma^2 + \sigma_A^2}{2}) + \frac{\sigma^2}{2} v''(a)(1-\beta)^2. \tag{31}$$

As stated in the proposition, aggregate risk is equally shared by the principal and the agent as in the first best (i.e., $\beta^A = 1$). This is because aggregate risk is publicly observable. So there is nothing to be gained by deviating from the first-best risk aggregate risk-sharing policy.

In spite of the simplification induced by homogeneity, equation (31) remains a complex second-order differential equation. It can only be solved numerically. Appendix A.2 explains how to obtain such a solution.

Taking the first order conditions in the Hamilton Jacobi Bellman equation (31), yields the optimal controls as a function of the state variable a_t:

$$\gamma_t = \gamma(a_t) = \frac{\rho}{1 - v'(a_t)} \tag{32}$$

and, for $\beta_t = \beta(a_t)$,

$$\rho v'(a_t)\beta_t + \rho(1 - v'(a_t)) + \sigma^2 v''(a_t)\beta_t^2(\beta_t - 1) = 0. \tag{33}$$

Moreover, the dynamics of the state variable a_t are

$$da_t = \Big[\rho(a_t - \log \frac{\rho}{\beta_t}) - (\mu - \gamma_t - \frac{\rho}{\beta_t} - \frac{\sigma^2 + \sigma_A^2}{2})\Big]dt + \sigma(\beta_t - 1)dZ_t \tag{34}$$

with initial condition $a_0 = a$.

Together, equations (32) to (34) yield the dynamics of the state variable and the controls in the optimal contract for one agent. As noted, we are not able to solve analytically for these dynamics, but the analysis implies that γ_t and β_t fluctuate randomly over time, which we exhibit in our simulations in Appendix A.2. As shown below, this contrasts with the case of a continuum of agents.

6 The case of a continuum of agents

We now consider the limit case of a continuum of agents by letting N go to ∞. While this restores tractability for linear compensation mechanisms, this also makes the problem mathematically challenging because it amounts to a control problem in an infinite dimensional space. This section first clarifies the

definition of a linear compensation mechanism, then sketches the mean-field limit of the problem, and finally presents the functional form of the Hamilton Jacobi Bellman equation in this context. We try to keep the main text simple and intuitive. In an online Appendix we offer an introduction to mean-field theory and explain how we use it in the context of the continuum limit of our model.

6.1 The mean-field limit

With a finite number of agents, the value function of the principal is a function of the aggregate capital, K_t, and the empirical distribution of continuation utilities, $\mathbb{W}$. In the continuum economy, the value function of the principal is a function of the aggregate capital K_t and the conditional law $\mathbb{P}_t$ of agents' continuation utility given the aggregate shocks up to time t.

Solving for the optimal mechanism in the continuum economy is in general a challenging problem, but can be achieved if the analysis is restricted to the subclass of linear mechanisms in which agents' consumption is proportional to the assets under their management. Formally, we have:

Definition. $X \equiv \left(c^P, \varphi, k, \beta^A, \beta\right)$ *is a linear mechanism if the ratio of consumption over capital under management only depends on aggregate capital* K *and the distribution of continuation utilities* $\mathbb{P}$ *but not on the individual continuation utility* ω*:*

$$\frac{c(K, \mathbb{P}, \omega)}{k(K, \mathbb{P}, \omega)} = \varphi(K, \mathbb{P}), \text{ for all } \omega. \tag{35}$$

In online Appendix B, we review in what sense the continuum economy is the limit of N-agent economies. In particular, for linear mechanisms as defined above, the aggregate consumption of agents at date t converges to

$$c_t^A := c^A(K_t, \mathbb{P}_t) := \int c(K_t, \mathbb{P}_t, \omega) \mathrm{d}\mathbb{P}_t(\omega), \tag{36}$$

while the resource constraint (5) converges to

$$dK_t = \left(\mu K_t - c^A(K_t, \mathbb{P}_t) - c^P(K_t, \mathbb{P}_t)\right) dt + \sigma_A K_t dZ_t^A. \tag{37}$$

This reflects that, when the number of agents goes to infinity, idiosyncratic noise diversifies, so that the only stochastic term in the dynamics of aggregate capital is the aggregate shock dZ_t^A. Similarly, the dynamics of an agent's continuation utility only contains two stochastic terms, the aggregate shock dZ_t^A and the agent's idiosyncratic shock dZ_t:[18]

$$d\omega_t = \rho\left(\omega_t - u(c_t)\right) dt + \sigma\beta dZ_t + \sigma_A \beta^A dZ_t^A. \tag{38}$$

[18] In line with Holmström's (1979) informativeness principle, it would be suboptimal to make ω_t^i depend on the idiosyncratic shocks of other agents, about which the agent does not have any private information.

The principal's control problem is

$$V(K,\mathbb{P}) = \sup_X \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt\right],$$

where the expectation is taken over aggregate shocks (Z_t^A), the state equations are given by the resource constraint (37) and the promise keeping condition (38), and maximization is taken over the set of controls $X \equiv (c^P(K,\mathbb{P}), k(K,\mathbb{P},\omega), \varphi(K,\mathbb{P}), \beta_A(K,\mathbb{P},\omega), \beta(K,\mathbb{P},\omega))$ that for all t satisfy the incentive compatibility condition

$$\beta(\omega) \geq \rho k(\omega) u'(\varphi k(\omega)), \tag{39}$$

the capital allocation constraint

$$K_t = \int_{\mathbb{R}} k(K_t, \mathbb{P}_t, \omega)\, \mathrm{d}\mathbb{P}_t(\omega). \tag{40}$$

and some regularity constraints that we spell out formally in Appendix A.3. By (35) and (36), we have $c^A = \varphi K$, so that (37) becomes

$$dK_t = \big((\mu - \varphi_t)K_t - c_t^P\big)dt + \sigma_A K_t\, dZ_t^A,$$

and the drift of (38) is $\rho\big(\omega_t - u(\varphi_t k_t(\omega_t))\big)$.

Compensation and capital allocation are the two instruments of the principal: the first sets the agents' aggregate consumption, the second distributes it across agents.

6.2 The functional Hamilton Jacobi Bellman equation

To analyze the optimal linear mechanism with a continuum of agents we must characterize the Hamilton Jacobi equation. To do so we must apply Itô's formula to the principal's value function V. But since V depends on the distribution of continuation utilities $\mathbb{P}$, this requires differential calculus in a space of probability measures. This generalization of classical calculus is achieved by first and second variations along mixtures of probability measures (linear functional derivatives), which we review in online Appendix B. First and second variations extend the classical notions of gradient (∇V) and Hessian $(\nabla^2 V)$ to functionals defined on a set of probability measures. Using these tools we write the Hamilton Jacobi Bellman equation in the continuum of agents case as follows:

$$\rho V(K,\mathbb{P}) = \sup_{c^P,\varphi,k,\beta^A,\beta} \Big\{ \rho u(c^P) + V_K\left((\mu-\varphi)K - c^P\right)$$
$$+\frac{1}{2}V_{KK}\sigma_A^2 K^2 + \int \partial_\omega \nabla V(\omega)\rho(\omega - u(\varphi k(\omega)))\mathrm{d}\mathbb{P}(\omega)$$

$$+\frac{1}{2}\int \partial^2_{\omega\omega}\nabla V(\omega)\left(\sigma^2\beta(\omega)^2+\sigma_A^2\beta^A(\omega)^2\right)\mathrm{d}\mathbb{P}(\omega)+\int \partial^2_{\omega K}\nabla V(\omega)\sigma_A^2\beta^A(\omega)K\mathrm{d}\mathbb{P}(\omega)$$
$$+\frac{1}{2}\int\int \partial^2_{\omega\omega'}\nabla^2 V(\omega,\omega')\sigma_A^2\beta^A(\omega)\beta^A(\omega')\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}(\omega')\Bigg\}, \tag{41}$$

where the supremum in (41) is subject to the capital allocation constraint (40) and the incentive compatibility condition (39). The capital allocation constraint (40) links the controls $k(\cdot)$ to the state variables $(K,\mathbb{P})$. It is a state-dependent constraint on the set of admissible controls, and in our formal definition of admissibility in Appendix A.3, we impose it directly. The verification theorem in Proposition 9 below is stated for this constrained problem.

When $\partial^2_{\omega\omega}\nabla V(\omega,K)<0$ (which will be checked ex post), the incentive compatibility condition optimally binds. This yields the following condition, which links the three controls β, k, and φ :

$$\beta(\omega)=\rho k(\omega)u'(\varphi k(\omega))>0. \tag{42}$$

The intuition is that because of risk aversion, constrained optimality implies that the sensitivity of each agent's continuation utility to its current performance is exactly what is needed for incentive compatibility.

6.3 The symmetric information case

To provide a benchmark, we end this section by stating the optimal linear mechanism with a continuum of agents under symmetric information. As can be seen in the following proposition, it closely resembles its finite number of agent counterpart. In particular, we generalize the aggregate inverse utility of agents (13) to

$$A_t=\int exp(\omega)\mathrm{d}\mathbb{P}_t(\omega). \tag{43}$$

Proposition 5. *When u is logarithmic and there is a continuum of agents, under symmetric information the optimal linear mechanism exists if and only if*

$$A=\int exp(\omega)\mathrm{d}\mathbb{P}(\omega)<m_\infty K,$$

where $m_\infty=\rho\exp\frac{1}{\rho}(\mu-\rho-\frac{\sigma_A^2}{2})$. The value function of the principal is given by (15) replacing m_N by its limit m_∞. The optimal mechanism has the following features for all t:

- *The ratio A_t/K_t is constant over time.*
- *The consumption flow of the principal is proportional to the aggregate capital stock: $c_t^P=\gamma K_t$.*
- *The compensation rate of the agents is $\varphi^{FB}=\frac{\rho A}{m_\infty K}=\rho-\gamma$.*
- *The capital allocation is given by $k_t=e^{\omega_t}\frac{K_t}{A_t}$.*

- *Aggregate capital and individual capital follow a geometric Brownian motion:*
$$\frac{dK_t}{K_t} = \frac{dk_t}{k_t} = (\mu - \rho)\, dt + \sigma_A dZ_t^A.$$
- *Aggregate risk is equally shared, while idiosyncratic shocks are fully insured:* $\beta_t^A = 1$ *and* $\beta_t = 0$.

7 An explicit solution when utility is logarithmic

From now on, we assume that $u(c) = \log c$. By analogy with the first best, we guess (and then verify) that the value function of the principal only depends on two scalars, aggregate capital K_t and the exhaustive statistic A_t for $\mathbb{P}_t$ from (43).

Similarly to the one-agent case (see (29) and (30)), we guess that the value function can be written as

$$V(K_t, \mathbb{P}_t) = \log K_t + v(a_t), \tag{44}$$

where

$$a_t := \log \frac{A_t}{K_t} \tag{45}$$

and v is a twice differentiable function that is to be determined.

Under our guess, the partial derivatives of V with respect to K are the same as in the case $N = 1$.[19] We can now also compute the first variation of the value function (44) and its derivatives. First, the gradient of V with respect to $\mathbb{P}$ is proportional to $\exp\omega$:

$$\nabla V(\omega) = \frac{v'(a)\exp\omega}{A}. \tag{46}$$

This implies

$$\partial_\omega \nabla V = \partial^2_{\omega\omega} \nabla V = \nabla V, \tag{47}$$

and

$$\partial_{\omega K} \nabla V = -\frac{v''(a)\exp\omega}{KA}. \tag{48}$$

Finally,

$$\nabla^2 V(\omega, \omega') = \frac{v''(a)\exp\omega\exp\omega'}{A^2} - \frac{v'(a)\exp\omega\exp\omega'}{A^2} = \partial^2_{\omega\omega'} \nabla^2 V(\omega, \omega'). \tag{49}$$

Setting $k(\omega) = \kappa(\omega)\frac{K}{A}\exp(\omega)$, the HJB equation (41) can then be written as

$$\begin{aligned}\rho v(a) = \sup_{\gamma,\varphi,\kappa,\beta,\beta^A} \Big\{ & \rho \log\gamma + \big(1 - v'(a)\big)\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2}\Big) \\ & + \rho\, v'(a)\Big(a - \log\varphi - \frac{1}{A}\int \log\kappa\, e^\omega \mathrm{d}\mathbb{P}\Big) \\ & + \frac{\sigma^2}{2} v'(a) \int \beta^2\, e^\omega \mathrm{d}\mathbb{P}/A + \frac{\sigma_A^2}{2}\Big[v'(a)\Big(\int \beta^{A^2}\, e^\omega \mathrm{d}\mathbb{P}/A - b^2\Big) + v''(a)(b-1)^2\Big]\Big\},\end{aligned} \tag{50}$$

[19] They are given in (78) in Appendix A.2.

where $b = \frac{1}{A}\int \beta^A \, e^{\omega} \mathrm{d}\mathbb{P}$, and the supremum is taken over $\gamma > 0$, $\varphi > 0$ and functions $\kappa > 0$, β such that $\int \kappa \, \mathrm{d}\tilde{\mathbb{P}} = 1$ and $\beta(\omega) \geq \frac{\rho}{\varphi}$ for all ω.

7.1 Optimal policy under the guess

The next proposition (whose proof is in Appendix A.3) provides the key characterization of optimal linear mechanisms.

Proposition 6. *Assume that there is a solution of the HJB equation* (41) *of the form* $V = \log K + v(a)$, *with* $v' < 0$ *and* $v'' < 0$. *The optimal linear mechanism has the following properties:*

- *The principal consumes a fraction of aggregate capital, that only depends on* $a \equiv \log \frac{A}{K}$*:*

$$c^P(K, \mathbb{P}) = \gamma(a)K = \frac{\rho}{1 - v'(a)}K. \tag{51}$$

- *The capital allocated to an agent with continuation utility* ω *is proportional to* $\exp \omega$*:*

$$k(K, \mathbb{P}, \omega) = \exp(\omega - a). \tag{52}$$

- *Each agent consumes a fraction* φ *of its assets under management such that*

$$\frac{\sigma^2 \rho^2}{\varphi^3} + \frac{\rho}{\varphi} = 1 - \frac{1}{v'(a)}. \tag{53}$$

- *The exposure to idiosyncratic risk is the same for all agents:* $\beta = \frac{\rho}{\varphi}$.

- *The exposure to aggregate risk is constant and the same for all agents, i.e.,* $\beta^A \equiv 1$, *as in the first-best.*

Note that the optimal exposure to aggregate risk is the same in the second-best as in the first-best because aggregate risk is publicly observable and can therefore be shared optimally without jeopardizing incentives. Equation (51) states that the consumption of the principal is proportional to aggregate capital. The proportionality factor depends on the state variables (aggregate capital and distribution of utilities) only via the aggregator a.

7.2 Solution of the HJB equation under the guess

Under the guess, the HJB equation (41) simplifies to

$$\begin{aligned} v(a) = \sup_{\varphi > 0} \Big(\log \frac{\rho}{1 - v'(a)} + \frac{1}{\rho}\big(\mu - \varphi - \rho - \frac{\sigma_A^2}{2}\big) \\ + v'(a)\Big(a - \log \varphi - \frac{1}{\rho}\big(\mu - \varphi - \frac{\sigma_A^2}{2} - \frac{\sigma^2 \rho^2}{2\varphi^2}\big)\Big)\Big). \end{aligned} \tag{54}$$

The Hamilton Jacobi Bellman equation (54) is simpler than its counterpart with one agent (31). With a continuum of agents, the equation is of order 1, while it is of order 2 with one agent. The differential equation (54) is equivalent to a simpler static optimization problem.

Proposition 7. *The domain of definition of the solution of the Hamilton Jacobi Bellman equation (54) is* $[0, a_{max}]$*, where*

$$a_{max} := \sup_{\varphi>0} \big(\log \varphi + \frac{1}{\rho}(\mu - \varphi - \frac{\sigma_A^2}{2} - \frac{\sigma^2\rho^2}{2\varphi^2})\big). \tag{55}$$

On this domain, the function v *is the solution of the following optimization problem:*

$$v(a) = \sup_{\gamma>0,\ \varphi>0} \Big(\log \gamma + \frac{1}{\rho}\big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2}\big)\Big), \tag{56}$$

subject to

$$a = \log \varphi + \frac{1}{\rho}\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2} - \frac{\sigma^2\rho^2}{2\varphi^2}\Big). \tag{57}$$

The interpretation of Proposition 7 is natural. The objective function (56) of the reduced problem is equal to the principal's lifetime utility under constant controls when $K = 1$: log consumption plus the ratio of the expected growth rate minus a risk premium over the discount rate. The constraint (57) gives the value of $a = \log(A/K)$ when $K = 1$, that is, the logarithm of the average inverse utility $\int e^{\omega}\, \mathrm{d}\mathbb{P}$ of the agents.

The difference in risk-bearing between the principal and the agents stems from the fact that the principal is only exposed to the aggregate risk (as the aggregate impact of idiosyncratic risk on output diversifies away), while each agent is also exposed to his/her own idiosyncratic risk for incentive reasons.

The relation between the instantaneous optimization problem (56) and the HJB equation (54) is the following: the right hand side of (54) can be interpreted as the Lagrangian of (56) by noting that the Lagrange multiplier equals $v'(a)$, once the Lagrangian has been maximized in γ, which gives $\gamma = \rho/(1 - v'(a))$ as in (51).

Intuitively, there exists a solution to the optimization problem if the amount of capital available K is large enough to enable the principal to deliver the distribution of continuation utilities $\mathbb{P}$ (summarized by A), while retaining a positive consumption, i.e. $\gamma > 0$. This is possible when a is less than the value a_{max} that would obtain by setting $\gamma = 0$ in the constraint (57).

Note that when $\sigma = 0$, $v(a) = \log(m_\infty - \exp a)$, corresponding to the first best value function characterized in Proposition 5.

The strict concavity of v established in Proposition 7 implies an important property of the optimal linear mechanism stated in the next proposition.

Proposition 8. *With logarithmic utilities, in the optimal mechanism under our guess, the ratio of average inverse utility to aggregate capital,* $\frac{A_t}{K_t} \equiv e^a$*, is constant along optimal trajectories.*

The property stated in Proposition 8 further simplifies the problem and shows that in the optimal linear mechanism controls are constant along optimal trajectories.

The analysis up to now implies that, with logarithmic utilities, our initial guess (44) leads to the value function given in Proposition 7, which satisfies the HJB equation (41) within the class of admissible mechanisms defined formally in Appendix A.3. It remains to verify that this function indeed yields the principal's value function. This is done in the next Proposition.

Proposition 9. *Let $\mathcal{A}^{\varphi}(K,\mathbb{P})$ be the set of admissible mechanisms defined by conditions (a)–(d) in Appendix A.3, and let $a = \log(A/K) < a_{max}$, where $A = \int e^{\omega}\,\mathrm{d}\mathbb{P}(\omega)$. The function $V(K,\mathbb{P}) = \log K + v(a)$, with v given by Proposition 7, is an upper bound for the principal's payoff over $\mathcal{A}^{\varphi}(K,\mathbb{P})$. If, in addition, $\int e^{2\omega}\,\mathrm{d}\mathbb{P}(\omega) < \infty$, the controls identified in Proposition 6 belong to $\mathcal{A}^{\varphi}(K,\mathbb{P})$ and achieve this bound, so that $V(K,\mathbb{P})$ is the value of the control problem of the principal over $\mathcal{A}^{\varphi}(K,\mathbb{P})$.*

7.3 Consumption and exposure to idiosyncratic risk

Relying on the analysis of the previous section we now present intuitive properties of the optimal controls γ and β.

As noted in (51), $\gamma = \frac{\rho}{1-v'(a)}$. Furthermore, from Proposition 7,

$$\frac{\sigma^2}{\rho}\beta^3 + \beta = 1 - \frac{1}{v'(a)}.$$

We can eliminate $v'(a)$ between these two conditions, implying an increasing relation between γ and β:

$$\gamma = \rho\left(1 - \frac{1}{\frac{\sigma^2}{\rho}\beta^3 + \beta}\right). \tag{58}$$

Since v is strictly concave, we see that both γ and β are decreasing in a.

Proposition 10. *Both the consumption rate γ of the principal and the exposure of agents to their idiosyncratic risk β (resp. the compensation rate φ) decrease (resp. increases) with a.*

The interpretation of the proposition is the following. As the initial utility of agents increases, a increases. So, the larger a, the larger the utility agents can demand. To provide this utility, the principal must reduce his consumption and increase that of the agents, which reduces the agents' risk exposure. That is, managers with better outside options obtain a higher fee and face a lower performance sensitivity. Moreover, it can be shown that the compensation rate is larger than in the first best, i.e., for every $a < a_{max}$, $\varphi(a) > \varphi^{FB}(a) = \rho e^{a}/m_{\infty}$.

Proposition 10 is illustrated in Figure 1, which plots β as a function of a. To generate the figure, the expected productivity of the risky production technology, μ, is set at 15%, while the standard deviation of the idiosyncratic risk, σ, is set at 10%. Moreover, the agent discount rate, ρ, is set to 4%, while the standard deviation of the aggregate shock, σ_A, is set to 5%. The figure illustrates how, in the optimal mechanism, β decreases as a increases.

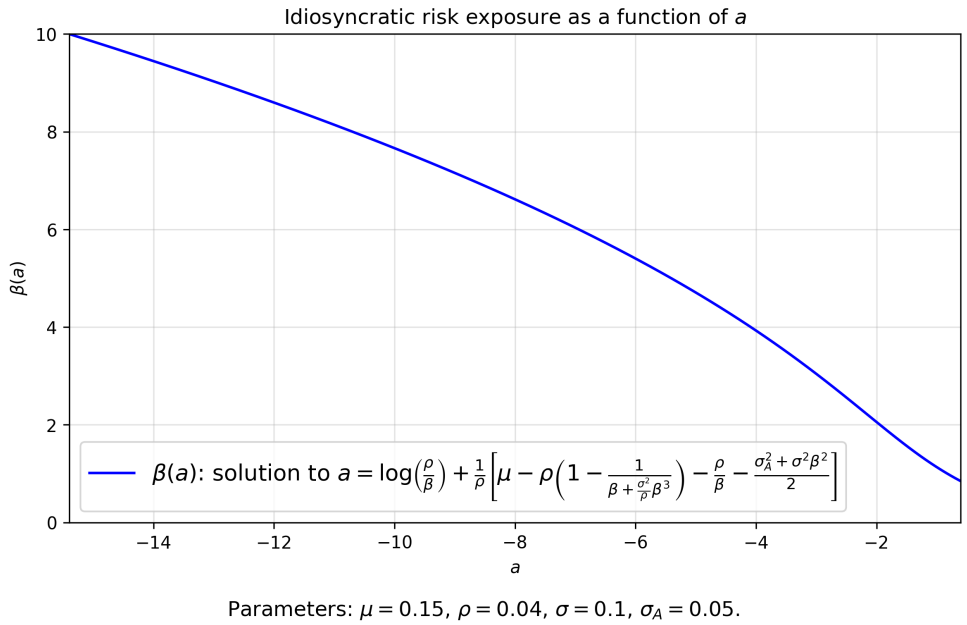


Figure 1: Idiosyncratic risk exposure as a function of bargaining power.

7.4 Dynamics of capital and utility

The next proposition characterizes the dynamics of capital in the optimal linear mechanism.

Proposition 11. *In the optimum (51)-(52), the dynamics of aggregate capital is*

$$\frac{dK_t}{K_t} = g_{SB}dt + \sigma_A dZ_t^A, \tag{59}$$

where the expected growth rate g_{SB} is given by

$$g_{SB} = \mu - \rho - \frac{\rho\sigma^2\varphi}{\varphi^2 + \rho\sigma^2}, \tag{60}$$

and the dynamics of individual capital is

$$\frac{dk_t}{k_t} = \frac{dK_t}{K_t} + \sigma\rho/\varphi dZ_t. \tag{61}$$

In contrast with the first best, with asymmetric information, condition (61) shows that capital allocation reflects incentives considerations. Thus, the capital of agents with above average performance grows faster than the capital of agents with below average performance. This can be interpreted as a capital allocation rule that rewards performance relative to the market benchmark, and punishes under-performance relative to that benchmark.

Equation (60) implies that when $\sigma = 0$, so that there is no incentive problem, the growth rate is equal to its first best level $(\mu - \rho)$. However,

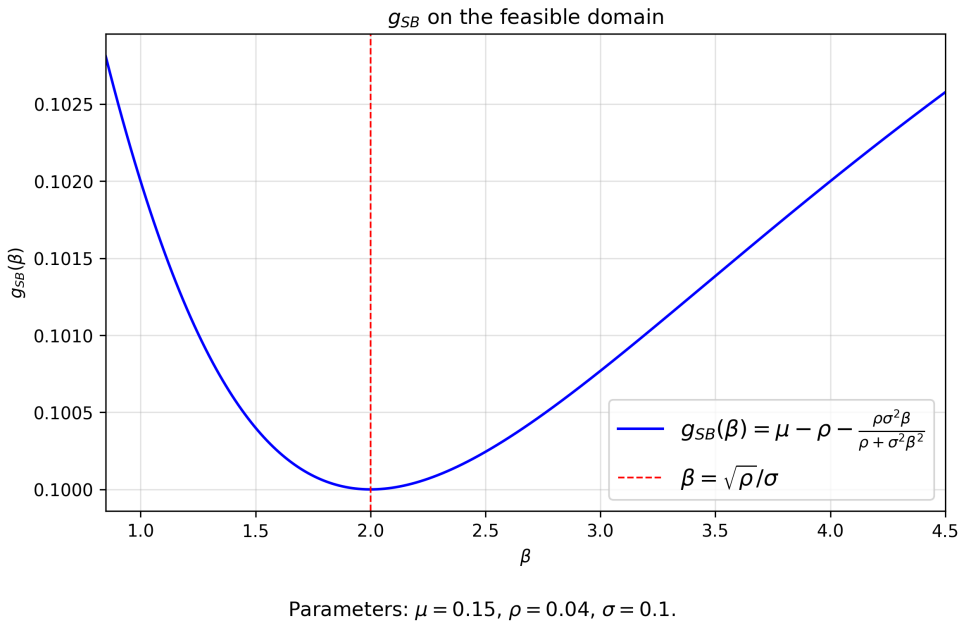


Figure 2: Expected growth as a function of idiosyncratic risk exposure.

when $\sigma > 0$, the expected growth is lower in the second best than in the first best. This is driven by the fact that incentive compatibility precludes perfect risk-sharing. To relax the incentive constraint, and thus improve risk-sharing, it is optimal to lower capital and raise consumption relative to the first best. This, in turn, reduces growth.

Equation (60) also implies that the aggregate expected growth rate g_{SB} is decreasing in β for $\beta < \sqrt{\rho}/\sigma$, and increasing for $\beta \geq \sqrt{\rho}/\sigma$. This is illustrated in Figure 2, which plots expected growth as a function of β for the same parameter values as in Figure 1.

Expected growth is non-monotonic in β because an increase in β has two effects that go in opposite directions: On the one hand, other things equal, an increase in β reduces agents' consumption, which increases growth. On the other hand, an increase in β increases the consumption of the principal, which reduces growth. As can be seen in the figure, for large values of β, the first effect dominates the latter.

Our analysis also yields the optimal dynamics of continuation utility, which is stated in the next proposition.

Proposition 12. *Agents' optimal continuation utilities follow the drifted Brownian motion*

$$\omega_t = \omega_0 + \left(g_{SB} - \frac{\beta^2\sigma^2 + \sigma_A^2}{2} \right) t + \beta\sigma Z_t + \sigma_A Z_t^A. \tag{62}$$

Equation (62), illustrated in Figure 3, characterizes these dynamics for some initial value of a. Figure 3 is generated for the same parameter values regarding μ, ρ, and σ as Figure 1 and Figure 2.

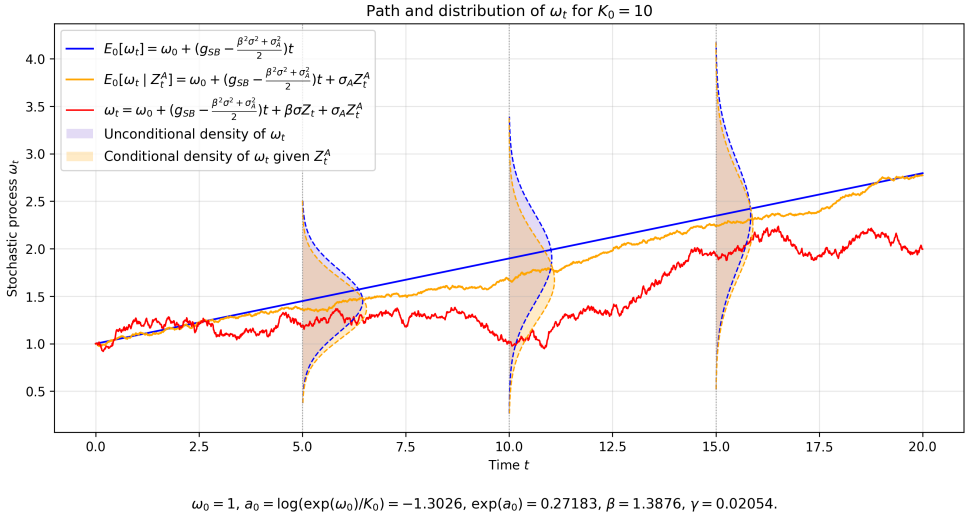


Figure 3: Dynamics of agents' continuation utility.

The blue line in Figure 3 is the expectation of the time t continuation utility of an agent conditional on time-0 information. This expectation is affine in t and equal to

$$\omega_0 + (g_{SB} - \frac{\beta^2\sigma^2 + \sigma_A^2}{2})t.$$

The slope reflects the opposite effects of expected aggregate growth, which tends to increase the utility of agents, and of costly risk bearing, which tends to reduce the utility of agents. If capital productivity μ is large enough, the positive effect of growth dominates the negative effect of costly risk-bearing, so that, in expectation, agent's utility tends to increase. This is the case for the parameters used to generate the figure. So, in the figure, the blue line is increasing.

The blue bell shape in the figure depicts the distribution of agents' time-t-continuation utility around its expectation conditional on time 0 information. The dispersion around the expectation reflects the weighted sum of the two Brownian terms

$$\beta\sigma Z_t + \sigma_A Z_t^A.$$

The standard deviation of this Brownian term increases with the square root of time, so that, in Figure 3, the width of the blue bell shape increases with time. Its variance $(\beta^2\sigma^2 + \sigma_A^2)t$ includes the aggregate shock, which shifts the continuation utilities of all agents by the same amount and is also present under symmetric information. Cross-sectional inequality corresponds to the orange bell shape described below, whose variance $\beta^2\sigma^2 t$ is due to the incentive compatibility constraint, which implies that agents' continuation utilities must react to their idiosyncratic performance.

Similarly to the first best, if the capital productivity (μ) is high enough,

so that the drift in ω_t is positive, all agents' utilities go to infinity.[20] The difference between the first and second best is that, in the latter, the productivity needed for the continuation utilities to go to infinity is greater, to compensate for the negative effect of exposure to the idiosyncratic risk.

The stochastic process in orange in the figure is the expected continuation utility of the agent at time t, conditional on one sample path for the realization of the aggregate shock Z_t^A, that is,

$$\omega_0 + (g_{SB} - \frac{\beta^2\sigma^2 + \sigma_A^2}{2})t + \sigma_A Z_t^A.$$

The orange bell shape here depicts the distribution of the continuation utility at time t around this conditional expectation. The dispersion around this conditional expectation reflects only the idiosyncratic risk, multiplied by the exposure of the agent to this risk, that is, $\beta\sigma Z_t$. The blue bell shape shows more dispersion than the orange bell shape, since the former reflects uncertainty about Z_A^t and Z_t, while the latter only reflects uncertainty about Z_t. However, similar to the blue bell shape, the width of the orange bell shape increases with time, reflecting the increasing cross-sectional inequality between agents.

Finally, the stochastic process in red is the realization of the stochastic process of an agent's continuation utility (ω_t), conditional on one sample path for aggregate risk (Z_A^t), and one sample path for the agent's idiosyncratic risk (Z_t). Thus, Figure 3 illustrates that there is a simple relation between the continuation utility of an agent at date t and its performance over $(0, t)$. The total productivity of the agent on $(0, t)$ is $\mu t + \sigma Z_t + \sigma^A Z_t^A$. When there is no aggregate risk, optimal compensation implies a simple, affine relation between continuation utility ω_t and this performance measure, similarly to Holmström and Milgrom (1987). With aggregate risk, optimal compensation is more complex as it must also take into account aggregate performance.

Our analysis of the distribution of continuation utilities, combined with our result that $k(\omega) = \exp(\omega - a)$, where a is a constant, implies that the distribution of capital across agents is lognormal when the initial distribution of ω_0 is degenerate (as in Figure 4) or Gaussian.

Denoting the variance of ω_0 by s_0^2, the cross-sectional variance of $\log k_t$ is $s_0^2 + \beta^2\sigma^2 t$, and that of k_t/K_t is $e^{s_0^2+\beta^2\sigma^2 t} - 1$. Thus, both increase with time. The cross-sectional variance of capital levels, $K_t^2(e^{s_0^2+\beta^2\sigma^2 t} - 1)$, need not increase, since K_t can decrease.

7.5 Information-constrained Pareto frontier

The above analysis yields a characterization of the information-constrained Pareto frontier in the space of inverse utility ($\exp\omega$). Substituting the ex-

[20] Intuitively, this is due to the fact that the expectation of the continuation utility increases linearly with time, while the standard deviation increases as the square root of time. When t goes to infinity, the linear increase dominates the square root increase.

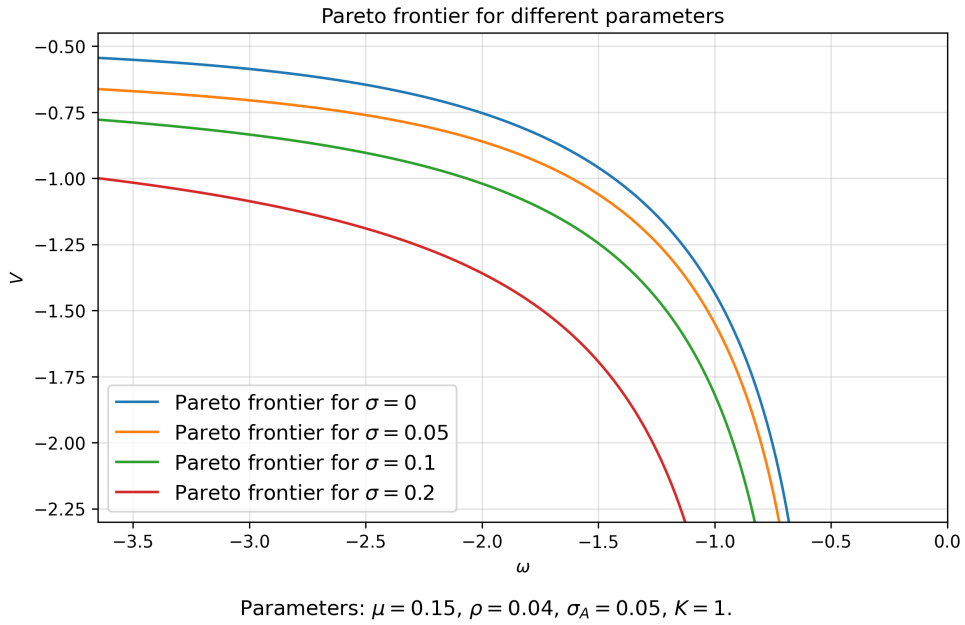


Figure 4: Pareto frontier for different values of σ.

pression of γ from (58) into the continuation value of the agent (62) and the optimal v identified in (56), we obtain

$$\exp \omega = \frac{\rho}{\beta} k(\omega) \exp\left[\frac{g_{SB}}{\rho} - \frac{\sigma^2\beta^2 + \sigma_A^2}{2\rho}\right], \tag{63}$$

where g_{SB} is the growth rate given in equation (60) and

$$\exp V(K, \mathbb{P}) = \rho K \left(1 - \frac{1}{\beta + \frac{\sigma^2}{\rho}\beta^3}\right) \exp\left[\frac{g_{SB} - \frac{\sigma_A^2}{2}}{\rho}\right]. \tag{64}$$

Figure 4 depicts the Pareto frontier generated by equations (63) and (64). In general, the Pareto frontier depends on the utilities of all agents. In order to draw a figure in two dimensions, we consider the case in which all agents start with one unit of capital, and thus all have the same initial utility. Thus, the horizontal axis is the representative agent's lifetime discounted expected (inverse) utility, while the vertical axis is the principal's lifetime discounted expected (inverse) utility. Different points on the Pareto frontier correspond to different values of β. As β increases, we move along the Pareto frontier to the northwest, corresponding to an allocation that is more favorable to the principal and less favorable to the agent.

The figure depicts this Pareto frontier for different values of σ. This shows that as the idiosyncratic risk increases, making the moral hazard problem more severe, the Pareto frontier shifts downwards.

8 Conclusion

This paper extends Merton's (1969) analysis of consumption-savings and asset-allocation choices to the case in which the investor delegates the management of assets to many, privately informed, agents. The characteristics of the optimal linear mechanism can be summarized as follows:

- Incentive compatibility implies that capital must be reallocated across agents as a function of performance, despite the fact that on the equilibrium path performance only reflects luck. Moreover, optimally capital reallocation is linear in relative performance.
- The optimal compensation rate is constant over time, higher than the rate that implements the first best, and inversely proportional to the sensitivity of continuation utility to performance.
- Exposing agents to their idiosyncratic risk increases the cost of holding risky capital and tilts the consumption-investment trade-off towards less investment. As a result, there is less capital accumulation than in the first best.
- Allocating capital as a function of performance leads to an increasingly heterogeneous distribution of capital among agents, which we characterize in quasi-closed form.

To obtain these results, we have considered a continuum of agents and taken a mean-field control approach to the investor's optimization problem. We thus develop a new methodology to solve dynamic contracting problems in which one the state variables is infinite-dimensional. This establishes a framework for a theory of large dynamic contracts, which complements the theory of large games. This new framework may be used to study incentive-constrained dynamic efficiency with heterogeneous agents in other contexts.

For example, our methodology could prove useful in dynamic macroeconomic models with heterogeneous agents and financial constraints (see Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Brunnermeier and Sannikov (2014), Huggett (1997), and Krusell and Smith (1998).) Accounting for aggregate risk has proved mathematically challenging in this literature. To cope with this difficulty, papers have often relied on numerical approximations. For example, Krusell and Smith (1998) assume that agents' expectations only depend on a one-dimensional statistics and then numerically calibrate this "approximated equilibrium." The tools developed in the present paper could help obtain exact and more explicit solutions. This could help shed more light on the key economic mechanisms at play in these problems.

Another example is incentive-constrained dynamic learning and innovation. Our new methodology, which makes it possible to study contracting with many agents could be fruitful in this context, for which transiting from one to many agents raises important new economic issues.

- For example, Manso (2011) studies the dynamic interaction between a principal and a single agent who can discover some innovation. Extending this analysis to the case of many agents would be particularly interesting in a Schumpeterian context, in which a population of agents can innovate as in Aghion and Howitt (1992). In that environment, when one agent innovates, this makes previous technologies obsolete, but also makes future innovations more attractive. Our methodology could help study how this tradeoff interacts with dynamic incentives, and the efficiency properties of the optimal contract.

- Moreover, DeMarzo and Sannikov (2017) and Hörner and Samuelson (2013), analyze dynamic principal-agent interactions with learning. The ability to compare the experiences of different agents could make learning more efficient, and at the same time affect the incentives of the agents. In this context, our methodology could prove useful to study the dynamic efficiency of incentive constrained allocations.

- Jenter and Lewellen (2021) study how CEO turnover interacts with incentives. In a model with many CEOs, CEO turnover could arise naturally, as better performing CEOs would be reallocated the assets of worse performing CEOs. Our methodology may shed light on the incentive-constrained efficiency of such dynamic reallocations.

While such analyses might be quite challenging, we hope that the methodological tools developed in this paper will help address these questions.

Appendix A: Proofs and Further Results

Appendix A.1. Symmetric Information

Proof of Proposition 2

Let $b = A/K$ and consider, on the domain $\mathcal{D} = \{(K, \mathbb{W}) : K > 0,\ A < m_N K\}$, the function

$$V(K, \mathbb{W}) = \log(m_N K - A).$$

The partial derivatives of V are

$$V_K = \frac{m_N}{K(m_N - b)},\ V_{KK} = -\frac{m_N^2}{K^2(m_N - b)^2},\ V_{\omega_i} = -\frac{\exp\omega_i}{NK(m_N - b)},$$
$$V_{\omega_i K} = \frac{m_N \exp\omega_i}{NK^2(m_N - b)^2},\ V_{\omega_i\omega_i} = -\frac{\exp 2\omega_i}{N^2K^2(m_N - b)^2} - \frac{\exp\omega_i}{NK(m_N - b)},$$
$$V_{\omega_i\omega_j} = -\frac{\exp\omega_i \exp\omega_j}{N^2K^2(m_N - b)^2} \quad (j \neq i).$$

The function V is the logarithm of the positive function $m_N K - A$, which is jointly concave in $(K, \mathbb{W})$ and thus V is concave. The diffusion coefficients of $(K, \mathbb{W})$ in (5) and (9) are linear in (k, β, β^A), and the terms in c^P and c^i are concave because $V_{\omega_i} < 0$. The expression inside the supremum in (11) is therefore concave in the controls so that the first-order conditions characterize the supremum. First-order conditions give

$$c^P = \rho\Big(1 - \frac{b}{m_N}\Big)K, \forall i, j,\ c^i = \frac{\rho\exp\omega_i}{m_N}, \quad k^i = K, \quad \beta^{ij} = \beta^{A,i} = 1.$$

Substituting these controls into (11) yields

$$0 = \frac{m_N}{m_N - b}\Big(\rho\log\frac{\rho}{m_N} + \mu - \rho - \frac{1}{2}\Big(\sigma_A^2 + \frac{\sigma^2}{N}\Big)\Big),$$

which holds by the definition (14) of m_N. Hence V solves the HJB equation (11) on $\mathcal{D}$. Along the controls, $c_t^P + \frac{1}{N}\sum_i c_t^i = \rho K_t$ and $k_t^i = K_t$ and thus K follows (21) with aggregate consumption rate ρ, which is (18). Moreover $\omega_t^i - \log c_t^i = \log(m_N/\rho)$, and the drift of $\omega_t^i - \log K_t$ is

$$\rho\log\frac{m_N}{\rho} - \Big(\mu - \rho - \frac{1}{2}\Big(\sigma_A^2 + \frac{\sigma^2}{N}\Big)\Big) = 0$$

by (14), while its diffusion terms cancel since $\beta^{ij} = \beta^{A,i} = 1$ and $k^i = K$. Hence $\exp(\omega_t^i)/K_t$ and b_t are constant, which gives (19) and (20) and consumption flows are proportional to K_t as in (17). It remains to show that no admissible mechanism gives the principal more than $\log(m_N K - A)$, and that no admissible mechanism gives her a finite payoff when $A \geq m_N K$. The argument is structurally the same as the one given at the end of the proof of Proposition 5 below and therefore omitted here. QED

Proof of Proposition 5

Consider a linear mechanism $X = (c^P, \varphi, k, \beta, \beta^A)$. Let $A_t = \int e^\omega \, \mathrm{d}\mathbb{P}_t(\omega)$ and $a_t = \log(A_t/K_t)$. To alleviate notation, we introduce the probability $\mathrm{d}\tilde{\mathbb{P}}_t = e^\omega \, \mathrm{d}\mathbb{P}_t/A_t$ and use the following change of variables in the controls: $\gamma_t = c_t^P/K_t$ and $k_t(\omega) = K_t \kappa_t(\omega) e^\omega / A_t$.

The capital allocation constraint (40) gives $\int \kappa_t \, \mathrm{d}\tilde{\mathbb{P}}_t = 1$.

We consider mechanisms in which the controls depend on $(K_t, \mathbb{P}_t)$ only through a_t, with $\kappa \equiv 1$:

$$\gamma_t = \gamma(a_t), \qquad c_t(\omega) = \varphi(a_t)\, k_t(\omega), \qquad \beta_t = \beta(a_t), \qquad \beta_{A,t} = \beta^A(a_t),$$

Note that, unlike under asymmetric information, under symmetric information, β is unrestricted. Then, $\omega - \log c_t(\omega) = a_t - \log \varphi(a_t)$ and (38) and (37) read

$$d\omega_t = \rho\big(a_t - \log \varphi(a_t)\big)dt + \sigma\beta(a_t)\, dZ_t + \sigma_A \beta^A(a_t)\, dZ_t^A \tag{65}$$

$$dK_t = \big(\mu - \varphi(a_t) - \gamma(a_t)\big) K_t \, dt + \sigma_A K_t \, dZ_t^A. \tag{66}$$

The coefficients of (65)–(66) depend on $(K_t, \mathbb{P}_t)$ only through a_t.

Integrating (65) gives

$$e^{\omega_t} = e^{\omega_0} \exp\Big[\rho \int_0^t \big(a_s - \log \varphi(a_s)\big) ds\Big] \exp\big(\sigma \Phi_t\big) \exp\Big(\sigma_A \int_0^t \beta^A(a_s)\, dZ_s^A\Big), \tag{67}$$

where

$$\Phi_t = \int_0^t \beta(a_s)\, dZ_s. \tag{68}$$

Remember that

$$A_t = \int e^\omega \, \mathrm{d}\mathbb{P}_t(\omega) = \mathbb{E}\big[e^{\omega_t} \,\big|\, \mathcal{F}_t^A\big].$$

Since a is $(\mathcal{F}_t^A)$-adapted and (ω_0, Z) is independent of Z^A, conditionally on $\mathcal{F}_t^A$ the variable Φ_t is centered Gaussian with variance $\int_0^t \beta(a_s)^2 \, ds$, and independent of ω_0. Hence

$$\mathbb{E}\big[e^{\sigma\Phi_t} \,\big|\, \mathcal{F}_t^A\big] = \exp\Big(\frac{\sigma^2}{2} \int_0^t \beta(a_s)^2 \, ds\Big).$$

The second and the fourth factors on the right-hand side of (67) are $\mathcal{F}_t^A$-measurable. Taking expectations conditional on $\mathcal{F}_t^A$ in (67) therefore yields, with $A_0 = \int e^\omega \, \mathrm{d}\mathbb{P}_0(\omega)$,

$$A_t = A_0 \exp\Big[\int_0^t \Big(\rho\big(a_s - \log \varphi(a_s)\big) + \frac{\sigma^2 \beta(a_s)^2}{2}\Big) ds\Big] \exp\Big(\sigma_A \int_0^t \beta^A(a_s)\, dZ_s^A\Big). \tag{69}$$

Integrating (66) yields

$$K_t = K_0 \exp\Big[\int_0^t \Big(\mu - \varphi(a_s) - \gamma(a_s) - \frac{\sigma_A^2}{2}\Big) ds + \sigma_A Z_t^A\Big]. \tag{70}$$

Dividing (69) by (70), we get

$$a_t = \log\frac{A_0}{K_0} + \int_0^t \Big(\rho\big(a_s - \log\varphi(a_s)\big) + \frac{\sigma^2\beta(a_s)^2}{2} - \mu + \varphi(a_s) + \gamma(a_s) + \frac{\sigma_A^2}{2}\Big)ds + \sigma_A\int_0^t \big(\beta^A(a_s) - 1\big)\,dZ_s^A. \tag{71}$$

For the optimal mechanism, $\gamma(a) = \rho/(1 - v'(a))$ and $\beta^A = 1$, so the martingale term in (71) vanishes. Differentiating (71) gives

$$\mathrm{d}a_t = \theta(a_t)\,\mathrm{d}t + \sigma_A(b_t - 1)\,\mathrm{d}Z_t^A$$

with

$$\theta(a) = \rho\big(a - \log\varphi(a)\big) + \frac{\sigma^2}{2}\beta(a)^2 + \frac{\sigma_A^2}{2} - \big(\mu - \gamma(a) - \varphi(a)\big). \tag{72}$$

Moreover, the evolution of capital is

$$\frac{\mathrm{d}K_t}{K_t} = (\mu - \gamma_t - \varphi_t)\,\mathrm{d}t + \sigma_A\,\mathrm{d}Z_t^A.$$

For $V = \log K + v(a)$, the terms in $\log K$ cancel in the HJB equation (41) (without the incentive constraint under the supremum). If we let $b = \int \beta^A\,\mathrm{d}\tilde{\mathbb{P}}$, the HJB reads

$$\rho v(a) = \sup_{\gamma,\varphi,\kappa,\beta,\beta^A} \Big\{\rho\log\gamma + \mu - \gamma - \varphi - \frac{\sigma_A^2}{2} + \theta\,v'(a) + \frac{\sigma_A^2}{2}(b-1)^2 v''(a)\Big\}, \tag{73}$$

where

$$\theta = \rho\Big(a - \log\varphi - \int\log\kappa\,\mathrm{d}\tilde{\mathbb{P}}\Big) + \frac{\sigma^2}{2}\int\beta^2\,\mathrm{d}\tilde{\mathbb{P}} + \frac{\sigma_A^2}{2}\Big(\int{\beta^A}^2\,\mathrm{d}\tilde{\mathbb{P}} - b^2 + 1\Big) - (\mu - \gamma - \varphi),$$

and the supremum is taken over $\gamma > 0$, $\varphi > 0$, $\kappa > 0$ with $\int\kappa\,\mathrm{d}\tilde{\mathbb{P}} = 1$, and unrestricted β and β^A.

We will now check that $v(a) = \log(m_\infty - e^a)$ for $a < \log m_\infty$ satisfies (73). The function v is C^2, bounded above by $\log m_\infty$, and satisfies $v'(a) = -e^a/(m_\infty - e^a) < 0$ and $v''(a) = -m_\infty e^a/(m_\infty - e^a)^2 < 0$. In (73), the control β enters only through $\frac{\sigma^2}{2}v'(a)\int\beta^2\,\mathrm{d}\tilde{\mathbb{P}}$, which is maximal at $\beta \equiv 0$. The control β^A enters only through $\frac{\sigma_A^2}{2}\big[v'(a)\big(\int{\beta^A}^2\,\mathrm{d}\tilde{\mathbb{P}} - b^2\big) + v''(a)(b-1)^2\big]$, which is non-positive by Jensen's inequality and maximal at $\beta^A \equiv 1$. The allocation enters only through $-\rho v'(a)\int\log\kappa\,\mathrm{d}\tilde{\mathbb{P}}$; since $-\rho v'(a) > 0$ and $\int\kappa\,\mathrm{d}\tilde{\mathbb{P}} = 1$, Jensen's inequality shows that it is maximal at $\kappa \equiv 1$, that is, $k(\omega) = Ke^\omega/A$. The terms in γ, $\rho\log\gamma - (1 - v'(a))\gamma$, are maximal at $\gamma = \rho/(1 - v'(a)) = \rho(1 - e^a/m_\infty)$, and the terms in φ, $-\rho v'(a)\log\varphi - (1 - v'(a))\varphi$, are maximal at $\varphi = -\rho v'(a)/(1 - v'(a)) = \rho e^a/m_\infty$. At these controls, $c = \varphi k = \rho e^\omega/m_\infty$, $a - \log\varphi = \log\frac{m_\infty}{\rho}$, $\gamma + \varphi = \rho$ and $b = 1$, so that $\theta = \rho\log\frac{m_\infty}{\rho} + \frac{\sigma_A^2}{2} - (\mu - \rho) = 0$

by the definition of m_∞, and the right-hand side of (73) equals $\rho \log \gamma + \mu - \rho - \frac{\sigma_A^2}{2} = \rho \log(m_\infty - e^a) = \rho v(a)$. Along this mechanism $a_t \equiv a$, so that the compensation rate $\varphi = \rho e^a / m_\infty$ is constant, which gives the properties listed in the proposition.

It remains to verify optimality. For a linear mechanism, $\log c_t^P = \log \gamma_t + \log K_t$ and Fubini's theorem give

$$\mathbb{E} \int_0^\infty \rho e^{-\rho t} \log c_t^P \, \mathrm{d}t = \log K + \mathbb{E} \int_0^\infty e^{-\rho t} r_t \, \mathrm{d}t, \tag{74}$$

$$r_t = \rho \log \gamma_t + \mu - \gamma_t - \varphi_t - \frac{\sigma_A^2}{2}. \tag{75}$$

We show that no mechanism gives the principal more than $\log(m_\infty K - A)$, without restricting the trajectories of a_t. Let X be a mechanism with aggregate consumption $C_t = c_t^P + \int c_t \, \mathrm{d}\mathbb{P}_t$ and $\Gamma_t = C_t / K_t$, such that

$$\mathbb{E} \int_0^\infty e^{-\rho t} (|\log \Gamma_t| + \Gamma_t) \, \mathrm{d}t < \infty$$

and

$$\mathbb{E} \int_0^\infty \rho e^{-\rho t} |\log c_t^P| \, \mathrm{d}t < \infty,$$

and such that each agent obtains his promised utility, $\mathbb{E}\big[\int_0^\infty \rho e^{-\rho t} \log c_t \, \mathrm{d}t \,\big|\, \omega_0 \big] = \omega_0$, with $\mathbb{E}\big[e^{\omega_0} \int_0^\infty \rho e^{-\rho t} |\log c_t| \, \mathrm{d}t \big] < \infty$. Fix $\ell > 0$ and set $\lambda = \ell e^{\omega_0}$. Since ω_0 is independent of Z^A, $\mathbb{E}[\lambda | \mathcal{F}_t^A] = \ell A =: \Lambda$ and $\mathbb{E}[\lambda \log \lambda | \mathcal{F}_t^A] = \ell A \log \ell + \ell \int \omega e^\omega \, \mathrm{d}\mathbb{P}(\omega)$. For $x, y > 0$, $\log x \le \log y + x/y - 1$. We apply this inequality to $x = c_t^P$, $y = C_t/(1+\Lambda)$, and to $x = c_t$, $y = \lambda C_t / (1+\Lambda)$, multiply the second inequality by λ, add the two inequalities and take expectations conditional on $\mathcal{F}_t^A$. Since $c_t^P + \mathbb{E}[c_t | \mathcal{F}_t^A] = C_t$, we obtain

$$\log c_t^P + \mathbb{E}[\lambda \log c_t | \mathcal{F}_t^A] \le (1+\Lambda) \log \frac{C_t}{1+\Lambda} + \ell A \log \ell + \ell \int \omega e^\omega \, \mathrm{d}\mathbb{P}(\omega). \tag{76}$$

Moreover,

$$\log K_t = \log K + \int_0^t (\mu - \Gamma_s - \frac{\sigma_A^2}{2}) \, \mathrm{d}s + \sigma_A Z_t^A$$

, so that, by Fubini's theorem and $\rho \log \Gamma - \Gamma \le \rho \log \rho - \rho$,

$$\begin{aligned} \mathbb{E} \int_0^\infty \rho e^{-\rho t} \log C_t \, \mathrm{d}t &= \log K + \mathbb{E} \int_0^\infty e^{-\rho t} \Big(\rho \log \Gamma_t + \mu - \Gamma_t - \frac{\sigma_A^2}{2} \Big) \mathrm{d}t \\ &\le \log K + \log \rho - 1 + \frac{\mu}{\rho} - \frac{\sigma_A^2}{2\rho} = \log(m_\infty K). \end{aligned}$$

By promise keeping, $\mathbb{E} \int_0^\infty \rho e^{-\rho t} \lambda \log c_t \, \mathrm{d}t = \ell \int \omega e^\omega \, \mathrm{d}\mathbb{P}(\omega)$. Multiplying (76) by $\rho e^{-\rho t}$, integrating and taking expectations, we obtain, for the principal's payoff J^X,

$$J^X \le (1 + \ell A) \log \frac{m_\infty K}{1 + \ell A} + \ell A \log \ell \qquad \text{for every } \ell > 0.$$

If $A < m_\infty K$, the choice $\ell = 1/(m_\infty K - A)$ gives $J^X \leq \log(m_\infty K - A)$. If $A \geq m_\infty K$, the right-hand side equals

$$\log \frac{m_\infty K}{1 + \ell A} + \ell A \log \frac{m_\infty K \ell}{1 + \ell A},$$

whose second term is negative and whose first term tends to $-\infty$ as $\ell \to \infty$, so that no mechanism gives the principal a finite payoff. This proves the upper bound and the necessity of the condition $A < m_\infty K$.

Along the guessed mechanism above, the supremum in (73) is attained at every date, $a_t \equiv a$ and $r_t \equiv \rho v(a)$, so that $\mathbb{E} \int_0^\infty e^{-\rho t} r_t \, \mathrm{d}t = v(a)$ and the payoff equals $\log(m_\infty K - A)$. The integrability condition holds along this mechanism because γ and φ are constant.

QED

Appendix A.2.
Asymmetric Information: The One Agent Case

Proof of Proposition 4

Formulas (29) and (30) imply that the partial derivatives of V are

$$V_\omega = v'(a), \quad V_{\omega\omega} = v''(a), \quad V_{\omega K} = -\frac{v''(a)}{K}, \tag{77}$$

$$V_K = \frac{1 - v'(a)}{K}, \quad V_{KK} = \frac{v''(a) + v'(a) - 1}{K^2}. \tag{78}$$

Write $\gamma = c^P/K$ and $\varphi = c/K$. After cancellation of $\rho \log K$, (26) becomes

$$\rho v(a) = \rho v'(a)\, a + \big(1 - v'(a)\big)\Big(\mu - \frac{\sigma^2 + \sigma_A^2}{2}\Big) \tag{79}$$

$$+ \sup_{\gamma > 0} \Big[\rho \log \gamma - \big(1 - v'(a)\big)\gamma\Big] + \sup_{\beta^A} \frac{\sigma_A^2}{2} v''(a)(\beta^A - 1)^2 + \sup_{\varphi, \beta \geq \rho} f(\varphi, \beta), \tag{80}$$

where the incentive constraint (27) reads $\varphi\beta \geq \rho$, with $\varphi > 0$ and $\beta > 0$, and

$$f(\varphi, \beta) = -\rho v'(a) \log \varphi - \big(1 - v'(a)\big)\varphi + \frac{\sigma^2}{2} v''(a)(\beta - 1)^2.$$

If $v'(a) \geq 1$, the supremum in γ is $+\infty$. If $v''(a) > 0$, the supremum in β^A is $+\infty$. Hence a finite right-hand side requires $v'(a) < 1$ and $v''(a) \leq 0$. When $v'(a) < 1$, the supremum in γ is attained at $\gamma = \rho/(1 - v'(a))$. If $v''(a) < 0$, the supremum in β^A is attained only at $\beta^A = 1$.

Hence, let $v'(a) < 1$ and $v''(a) < 0$. The feasible set $\{(\varphi, \beta) : \varphi > 0,\ \beta > 0,\ \varphi\beta \geq \rho\}$ is closed in $(0, \infty)^2$, and f tends to $-\infty$ along any feasible sequence that leaves every compact subset of $(0, \infty)^2$. Indeed, if $\varphi \to \infty$, the

term $-(1-v'(a))\varphi$ dominates $-\rho v'(a)\log\varphi$, and the β-term is nonpositive. If $\beta\to\infty$, the term $\frac{\sigma^2}{2}v''(a)\beta^2$ tends to $-\infty$. If $\varphi\to 0$, then $\beta\geq\rho/\varphi\to\infty$, and $\frac{\sigma^2}{2}v''(a)(\beta-1)^2$, of order $-\varphi^{-2}$, dominates $-\rho v'(a)\log\varphi$. If $\beta\to 0$, then $\varphi\geq\rho/\beta\to\infty$. Since f is continuous, the supremum of f on the feasible set is attained at some $(\hat{\varphi},\hat{\beta})$.

Suppose $\hat{\varphi}\hat{\beta}>\rho$. Then $(\hat{\varphi},\hat{\beta})$ is an interior point of the feasible set and a local maximum of f. Since $f(\varphi,\beta)$ is the sum of a function of φ and a function of β, $\partial_\beta f(\hat{\varphi},\hat{\beta})=\sigma^2 v''(a)(\hat{\beta}-1)=0$ gives $\hat{\beta}=1$, and $\partial_\varphi f(\hat{\varphi},\hat{\beta})=-\rho v'(a)/\hat{\varphi}-(1-v'(a))=0$ gives $v'(a)<0$ and $\hat{\varphi}=-\rho v'(a)/(1-v'(a))$. Then $\hat{\varphi}\hat{\beta}=\rho\,\frac{-v'(a)}{1-v'(a)}<\rho$, a contradiction. Hence $\hat{\varphi}\hat{\beta}=\rho$, the incentive constraint (27) binds, and $c=\rho K/\beta$.

Substituting $\beta^A=1$ and $\varphi=\rho/\beta$ into (79) and keeping the supremum over γ and β, now gives (31).

QED

Numerical analysis

The following lemma proved in the online appendix provides the precise boundary conditions that are needed for the numerical solution of the single-agent case in Section 5.

Lemma. *Let m_1 be given by (14) with $N=1$. The function v is finite on $(-\infty,\log m_1)$, the problem has no solution for $a\geq\log m_1$, and*

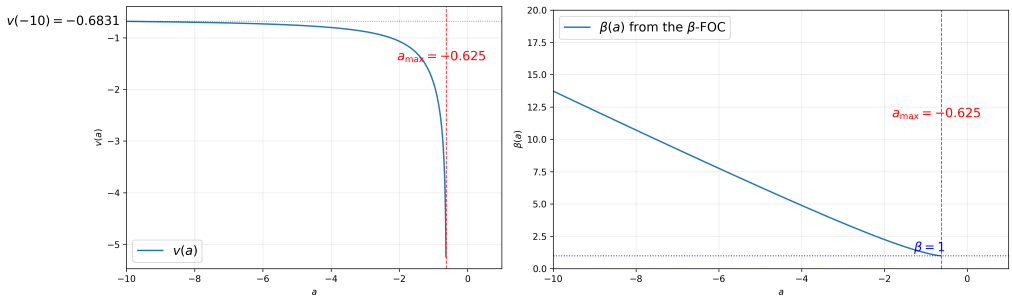
$$a+\log(\log m_1-a)\ \leq\ v(a)\ \leq\ \log(m_1-e^a)\qquad \textit{for all } a<\log m_1. \quad (81)$$

Consequently $\lim_{a\to\log m_1}v(a)=-\infty$. If moreover v is a concave, nonincreasing solution of (31), then $\lim_{a\to-\infty}v(a)=\log m_1$ and $\lim_{a\to-\infty}v'(a)=0$.

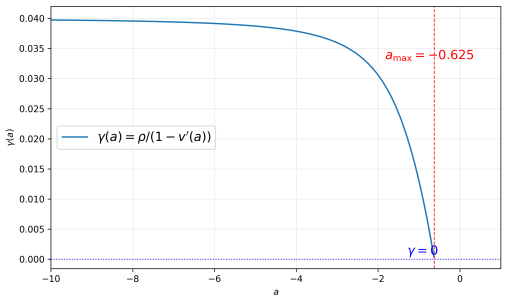
With these boundary conditions, we perform the numerical analysis using the `bvp` function from the SCIPY package. Figure 5, based on the numerical analysis, illustrates how the value function of the principal depends on the state variable a and it displays the curves until $a=-10$.[21] The figure also displays the associated control variables $\beta(a)$ and $\gamma(a)$ where $a_{\max}$ denotes the maximum value of a for which a solution exists.

The fluctuations of a_t in the single-agent case are illustrated in Figure 6, which plots two selected sample paths for a_t as well as the distribution of a_t after 5 periods, 10 periods, and 15 periods (estimated from 50000 simulated sample paths). The figure illustrates that, with a single agent, incentive-constrained optimal consumption and exposure to shocks undergo large random changes through time.

[21] The curves extend over all negative values of a.



(a) $v(a)$ (b) $\beta(a)$



(c) $\gamma(a)$

Figure 5: Illustration of the value function $v(a)$, and of the control variables $\beta(a)$ and $\gamma(a)$ for the parameters $K_0 = 100$, $\rho = 0.04$, $\mu = 0.15$, $\sigma = 0.1$, $\sigma_A = 0.05$ implying that $a_{\max} = -0.625$.

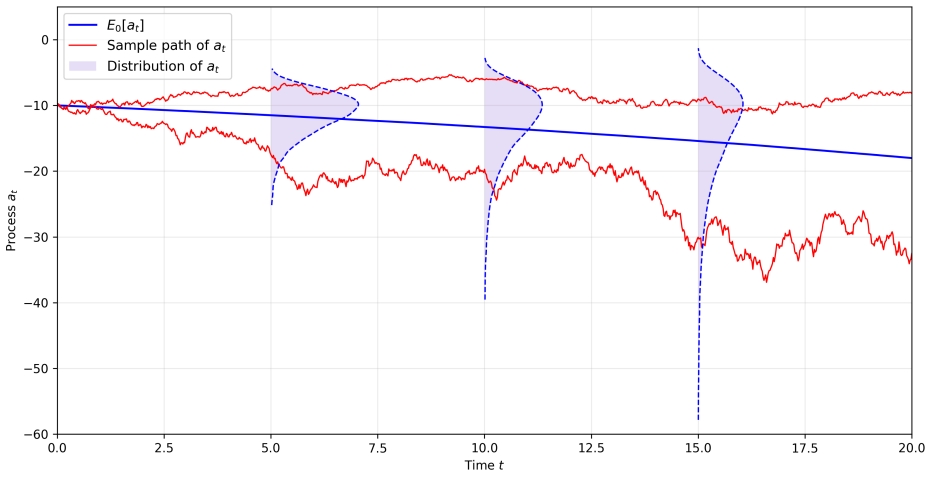


Figure 6: Dynamics of the state variable a_t for $\rho = 0.04$, $\mu = 0.15$, $\sigma = 0.1$, $\sigma_A = 0.05$ and $a_0 = -10$. The distributions are generated by 50000 samples. We use a root-finding algorithm to obtain β_t.

Appendix A.3.
Asymmetric Information: The Continuum Case

Proposition V below is the verification theorem for the problem of Section 7, i.e., for the principal's problem in the class of linear mechanisms. The capital allocation constraint is imposed directly on the admissible controls. With logarithmic utility, the state space is

$$\mathcal{E} = \{(K,\mathbb{P}) : K > 0,\ A = \int e^{\omega} \mathrm{d}\mathbb{P}(\omega) < \infty\}.$$

We write $\gamma_t = c_t^P / K_t$. We denote by $\mathcal{A}^{\varphi}(K,\mathbb{P})$ the set of mechanisms X with $(K_0^{(X)}, \mathbb{P}_0^{(X)}) = (K,\mathbb{P})$ such that:

(a) There is a process $\varphi_t > 0$, the same for all agents, such that $c_t(\omega) = \varphi_t k_t(\omega)$ with $k_t(\omega) > 0$, and $\beta_t(\omega) \geq \rho/\varphi_t$ for $\mathbb{P}_t$-a.e. ω and all t,

(b) $\int k_t \, \mathrm{d}\mathbb{P}_t = K_t$ for all t;

(c) $\mathbb{E}\int_0^{\infty} e^{-\rho t}\big(|\log\gamma_t| + \gamma_t + \varphi_t\big)\mathrm{d}t < \infty$, and for all T,

$$\mathbb{E}\int_0^T \int e^{2\omega}\big(\beta_t(\omega)^2 + \beta_{A,t}(\omega)^2\big)\mathrm{d}\mathbb{P}_t \, \mathrm{d}t < \infty,$$

and

$$\mathbb{E}\int_0^T \int e^{\omega}\big(|\omega - \log c_t(\omega)| + \beta_t(\omega)^2 + \beta_{A,t}(\omega)^2\big)\mathrm{d}\mathbb{P}_t \, \mathrm{d}t < \infty;$$

(d) there is $\varepsilon > 0$ such that $a_t \leq a_{max} - \varepsilon$ for all $t \geq 0$, almost surely.

Finally, define for any $X \in \mathcal{A}^{\varphi}(K,\mathbb{P})$ the gain function

$$J^{(X)}(K,\mathbb{P}) = \mathbb{E}\int_0^{\infty} \rho e^{-\rho t} u(c_t^P)\, dt.$$

The principal's value is

$$V(K,\mathbb{P}) = \sup_{X \in \mathcal{A}^{\varphi}(K,\mathbb{P})} J^{(X)}(K,\mathbb{P}).$$

The Hamilton Jacobi Bellman equation of the principal's problem is (41), where the supremum is taken over the set $\mathcal{U}^{\varphi}(K,\mathbb{P})$ of linear mechanisms.

Proposition V (Verification). *Let $v \in C^2((-\infty, a_{max}))$ be nonincreasing, and assume that, for every $(K,\mathbb{P}) \in \mathcal{E}$ with $a = \log(A/K) < a_{max}$, v satisfies (50), the supremum being taken over $\mathcal{U}^{\varphi}(K,\mathbb{P})$. Then, for every $(K,\mathbb{P}) \in \mathcal{E}$ with $a < a_{max}$,*

$$\sup_{X \in \mathcal{A}^{\varphi}(K,\mathbb{P})} J^{(X)}(K,\mathbb{P}) \ \leq\ \log K + v(a).$$

Proof of Proposition V. Let $X \in \mathcal{A}^\varphi(K,\mathbb{P})$ and $\varepsilon > 0$ as in (d). The first observation is that the HJB equation (50) is equivalent to

$$\rho v(a) = \sup_{\mathcal{U}^\varphi(K,\mathbb{P})} \Big\{ r + \theta\, v'(a) + \frac{\sigma_A^2}{2}(b-1)^2 v''(a) \Big\}, \tag{82}$$

where

$$r = \rho \log\gamma + \mu - \gamma - \varphi - \frac{\sigma_A^2}{2},$$

$$\theta = \rho\Big(a - \log\varphi - \int \log\kappa \,\mathrm{d}\tilde{\mathbb{P}}\Big) + \frac{\sigma^2}{2}\int \beta^2 \,\mathrm{d}\tilde{\mathbb{P}} + \frac{\sigma_A^2}{2}\Big(\int \beta^{A\,2}\,\mathrm{d}\tilde{\mathbb{P}} - b^2 + 1\Big) - (\mu - \gamma - \varphi).$$

Using (c), one can show that A_t is a continuous semimartingale with

$$\frac{\mathrm{d}A_t}{A_t} = \Big(\rho\Big(a_t - \log\varphi_t - \int \log\kappa_t \,\mathrm{d}\tilde{\mathbb{P}}_t\Big) + \frac{\sigma^2}{2}\int \beta_t^2 \,\mathrm{d}\tilde{\mathbb{P}}_t + \frac{\sigma_A^2}{2}\int \beta_{A,t}^2 \,\mathrm{d}\tilde{\mathbb{P}}_t\Big)\mathrm{d}t + \sigma_A b_t \,\mathrm{d}Z_t^A.$$

Itô's formula for $a_t = \log A_t - \log K_t$, with $\mathrm{d}K_t/K_t = (\mu - \gamma_t - \varphi_t)\mathrm{d}t + \sigma_A \mathrm{d}Z_t^A$, gives $\mathrm{d}a_t = \theta_t\,\mathrm{d}t + \sigma_A(b_t - 1)\,\mathrm{d}Z_t^A$, where θ_t and b_t are θ and b evaluated at the controls of X. By Itô's formula,

$$\begin{aligned}\mathrm{d}\big(e^{-\rho t}v(a_t)\big) = e^{-\rho t}\Big[&- \rho v(a_t) + \theta_t\, v'(a_t) + \frac{\sigma_A^2}{2}(b_t - 1)^2 v''(a_t)\Big]\mathrm{d}t \\ &+ e^{-\rho t}\sigma_A(b_t - 1)v'(a_t)\,\mathrm{d}Z_t^A.\end{aligned}$$

At each date, the values of X belong to $\mathcal{U}^\varphi(K_t,\mathbb{P}_t)$, and (82) is a supremum over this set, so that the bracket is at most $-r_t$. Let $\tau_n = \inf\{t \geq 0 : \int_0^t (b_s - 1)^2 v'(a_s)^2\,\mathrm{d}s \geq n\}$. Since v' is continuous on $(-\infty, a_{max})$ and $a_t \leq a_{max} - \varepsilon$, $\tau_n \to \infty$, and the stochastic integral stopped at τ_n is a martingale. Integrating between 0 and $T \wedge \tau_n$ and taking expectations,

$$v(a) \;\geq\; \mathbb{E}\int_0^{T\wedge\tau_n} e^{-\rho t} r_t \,\mathrm{d}t + \mathbb{E}\big[e^{-\rho(T\wedge\tau_n)} v(a_{T\wedge\tau_n})\big].$$

Since v is nonincreasing and $a_t \leq a_{max} - \varepsilon$, $v(a_t) \geq v(a_{max} - \varepsilon)$, so the last term is at least $-C_\varepsilon\,\mathbb{E}\big[e^{-\rho(T\wedge\tau_n)}\big]$ with $C_\varepsilon = \max(0, -v(a_{max} - \varepsilon))$. Since $\mathbb{E}\int_0^\infty e^{-\rho t}|r_t|\,\mathrm{d}t < \infty$ by (c), letting $n \to \infty$ and then $T \to \infty$ yields $v(a) \geq \mathbb{E}\int_0^\infty e^{-\rho t} r_t\,\mathrm{d}t$ by dominated convergence. Since $\log c_t^P = \log\gamma_t + \log K_t$ and $\log K_t = \log K + \int_0^t (\mu - \gamma_s - \varphi_s - \frac{\sigma_A^2}{2})\,\mathrm{d}s + \sigma_A Z_t^A$, Fubini's theorem and (c) give $J^{(X)}(K,\mathbb{P}) = \log K + \mathbb{E}\int_0^\infty e^{-\rho t} r_t\,\mathrm{d}t$, from which we deduce $J^{(X)}(K,\mathbb{P}) \leq \log K + v(a)$.

QED

Proof of Proposition 6

Under the guess (44)-(45), the HJB equation (41) reduces to (50). As before, let

$$c^P = \gamma K,\ c(\omega) = \varphi k(\omega),\ k(\omega) = K\kappa(\omega)\frac{e^\omega}{A} \text{ and } \mathrm{d}\tilde{\mathbb{P}} = \frac{e^\omega \mathrm{d}\mathbb{P}}{A}.$$

Then the capital allocation constraint reads $\int \kappa \, \mathrm{d}\tilde{\mathbb{P}} = 1$ and the incentive constraint is $\beta(\omega) \geq \rho/\varphi$ for $\tilde{\mathbb{P}}$-a.e. ω. Rearranging the terms under the sup in (50), we obtain

$$\begin{aligned}
&\big[\rho \log \gamma - (1-v')\gamma\big] + (1-v')\Big(\mu - \frac{\sigma_A^2}{2}\Big) + \rho a \, v' \\
&\quad - \big[(1-v')\varphi + \rho v' \log \varphi\big] - \rho v' \int \log \kappa \, \mathrm{d}\tilde{\mathbb{P}} + \frac{\sigma^2}{2} v' \int \beta^2 \, \mathrm{d}\tilde{\mathbb{P}} \\
&\quad + \frac{\sigma_A^2}{2}\Big[v'\Big(\int \beta^{A\,2} \, \mathrm{d}\tilde{\mathbb{P}} - b^2\Big) + v''(b-1)^2\Big],
\end{aligned}$$

(i) γ only appears in the first squared bracket. This has a maximum for $\gamma > 0$ iff $v' < 1$. In this case, the maximum is achieved by $\gamma = \frac{\rho}{1-v'}$, which gives (51).

(ii) If $v' > 0$ the maximand has no finite supremum because β can be made arbitrarily large. If $v' = 0$ any supremum cannot be attained, because $\varphi > 0$ can be made arbitrarily small. Hence, necessarily $v' < 0$. If $v'' > 0$ the third squared bracket can be made arbitrarily large by increasing β^A. Hence, necessarily $v'' \leq 0$.

(iii) By Jensen's inequality, $\int \beta^{A\,2} \, \mathrm{d}\tilde{\mathbb{P}} \geq b^2$, with equality if and only if β^A is constant. Hence, the last squared bracket is non-positive. If $v'' < 0$, it is zero iff $\beta^A \equiv 1$. If $v'' = 0$, β^A must be constant.

(iv) Since $\int \kappa \, \mathrm{d}\tilde{\mathbb{P}} = 1$, Jensen's inequality gives $\int \log \kappa \, \mathrm{d}\tilde{\mathbb{P}} \leq 0$, with equality if and only if $\kappa \equiv 1$. Since $v' < 0$, this is the unique maximizer.

v) Since $v' < 0$ and $\varphi > 0$, β must be pointwise minimized. Hence, the incentive constraint must bind and, as a consequence, φ is the unique maximizer of the concave function

$$-(1-v')\varphi - \rho v' \log \varphi + \frac{\sigma^2}{2} v' \frac{\rho^2}{\varphi^2}. \tag{83}$$

Collecting terms gives (54).

QED

Proof of Proposition 7

Define

$$f(\gamma,\varphi) = \log\gamma + \frac{1}{\rho}\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2}\Big), \qquad g(\gamma,\varphi) = \log\varphi + \frac{1}{\rho}\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2} - \frac{\sigma^2\rho^2}{2\varphi^2}\Big),$$

and $v(a) = \sup\{f(\gamma,\varphi) : \gamma > 0,\ \varphi > 0,\ g(\gamma,\varphi) = a\}$. We have

$$\partial_\varphi g(\gamma,\varphi) = \frac{\Delta(\varphi)}{\rho}, \qquad \partial_\varphi f(\gamma,\varphi) = -\frac{1}{\rho}$$

where

$$\Delta(\varphi) = \frac{\rho}{\varphi} - 1 + \frac{\sigma^2\rho^2}{\varphi^3} \text{ and } \Delta'(\varphi) = -\frac{\rho}{\varphi^2} - \frac{3\sigma^2\rho^2}{\varphi^4} < 0.$$

Since Δ is decreasing, $\Delta(\varphi) \to +\infty$ as $\varphi \to 0$ and $\Delta(\varphi) \to -1$ as $\varphi \to \infty$, there is a unique $\varphi_0 > 0$ with $\Delta(\varphi_0) = 0$. Hence $g(0, \cdot)$ increases on $(0, \varphi_0)$ and decreases on (φ_0, ∞), and $a_{max} = \sup_\varphi g(0, \varphi) = g(0, \varphi_0)$.
Since g is decreasing in γ, the constraint $g(\gamma, \varphi) = a$ has a solution $\gamma > 0$ if and only if $g(0, \varphi) > a$, and this solution is $\gamma_a(\varphi) = \rho\,(g(0, \varphi) - a)$. The constraint set is therefore non-empty if and only if $a < a_{max}$. For $a < a_{max}$, the set $B_a = \{\varphi > 0 : g(0, \varphi) > a\}$ is open, and its closure is a compact subset of $(0, \infty)$, because $g(0, \varphi) \to -\infty$ as $\varphi \to 0$ and as $\varphi \to \infty$. The function $\varphi \mapsto f(\gamma_a(\varphi), \varphi)$ is continuous on B_a and tends to $-\infty$ at the boundary of B_a, where $\gamma_a \to 0$. The supremum defining $v(a)$ is therefore attained.
Let $L(\gamma, \varphi, \nu) = f(\gamma, \varphi) + \nu\,(a - g(\gamma, \varphi))$. At a maximizer, the first-order conditions are

$$\frac{1}{\gamma} - \frac{1-\nu}{\rho} = 0, \qquad -\frac{1}{\rho}\big(1 + \nu\Delta(\varphi)\big) = 0,$$

that is,

$$\gamma = \frac{\rho}{1-\nu}, \qquad \nu = -\frac{1}{\Delta(\varphi)}, \qquad \text{hence} \qquad \gamma = \gamma(\varphi) := \frac{\rho\,\Delta(\varphi)}{\Delta(\varphi) + 1}.$$

The second condition reads $\nu\big(\frac{\sigma^2\rho^2}{\varphi^3} + \frac{\rho}{\varphi} - 1\big) = -1$, which is (53) with $\beta = \rho/\varphi$. Since $\Delta + 1 = \rho/\varphi + \sigma^2\rho^2/\varphi^3 > 0$, the condition $\gamma > 0$ requires $\Delta(\varphi) > 0$, that is, $\varphi < \varphi_0$. Hence $\nu = \nu(\varphi) := -1/\Delta(\varphi) < 0$.
For $\varphi \in (0, \varphi_0)$, let $G(\varphi) = g(\gamma(\varphi), \varphi)$. Then

$$\gamma'(\varphi) = \frac{\rho\,\Delta'(\varphi)}{(\Delta(\varphi) + 1)^2} < 0, \qquad G'(\varphi) = \partial_\varphi g + \partial_\gamma g\,\gamma'(\varphi) = \frac{\Delta(\varphi)}{\rho} - \frac{\gamma'(\varphi)}{\rho} > 0.$$

Moreover, $G(\varphi) \to g(0, \varphi_0) = a_{max}$ as $\varphi \to \varphi_0$, since $\gamma(\varphi_0) = 0$, and $G(\varphi) \to -\infty$ as $\varphi \to 0$, since $\gamma \le \rho$ and $g(0, \varphi) \to -\infty$. Hence G is a smooth increasing bijection from $(0, \varphi_0)$ onto $(-\infty, a_{max})$ and $G(\varphi(a)) = a$. The map $a \mapsto \varphi(a)$ is smooth and increasing.
Along the optimum, $v(G(\varphi)) = f(\gamma(\varphi), \varphi)$. The first-order conditions give $\nabla f = \nu \nabla g$, so that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} f(\gamma(\varphi), \varphi) = \nu(\varphi)\,\frac{\mathrm{d}}{\mathrm{d}\varphi} g(\gamma(\varphi), \varphi) = \nu(\varphi)\,G'(\varphi),$$

and therefore $v'(a) = \nu(\varphi(a))$. Fix $a < a_{max}$ and $\nu = v'(a) < 0$. The terms of L in γ are $\log\gamma - (1-\nu)\gamma/\rho$, maximal at $\gamma = \rho/(1-\nu)$ with value $\log\frac{\rho}{1-\nu} - 1$. After this maximization, $L(\cdot, \nu)$ coincides with the expression inside the supremum in (54). Its derivatives in φ are

$$\partial_\varphi L = -\frac{1}{\rho}\big(1 + \nu\Delta(\varphi)\big), \qquad \partial_\varphi^2 L = -\frac{\nu\,\Delta'(\varphi)}{\rho} < 0,$$

since $\nu < 0$ and $\Delta' < 0$. Hence $L(\cdot, \nu)$ is strictly concave in φ, and its unique critical point, $\varphi(a)$, is its global maximizer. At this point the constraint holds,

so that $L = f = v(a)$. This proves that v solves (54). Finally, the constraint $a = g(\gamma, \varphi)$ with $\gamma = \rho/(1 - v'(a))$ reads

$$\frac{\rho}{1 - v'(a)} + \rho\big(a - \log\varphi\big) + \varphi - \mu + \frac{1}{2}\Big(\sigma_A^2 + \frac{\sigma^2\rho^2}{\varphi^2}\Big) = 0,$$

which is (84).
$v'(a) = \nu(\varphi(a)) < 0$, so v is strictly decreasing. Since $\nu'(\varphi) = \Delta'(\varphi)/\Delta(\varphi)^2 < 0$ and $G'(\varphi) > 0$,

$$v''(a) = \frac{\nu'(\varphi(a))}{G'(\varphi(a))} < 0,$$

so v is C^2 and strictly concave.

QED

Proof of Proposition 8

Differentiating (54) along the optimal policy, using the Envelope Theorem and $v'' < 0$ yields

$$\frac{\rho}{1 - v'(a)} + \rho\big(a - \log\varphi\big) + \varphi - \mu + \frac{1}{2}\big(\sigma_A^2 + \frac{\sigma^2\rho^2}{\varphi^2}\big) = 0. \tag{84}$$

Note that Equation (84) is the constraint (57) of Proposition 7 with $\gamma = \rho/(1 - v'(a))$. Equation (84) yields Proposition 8. In fact, the right-hand side of equation (72) with $\beta^A = 1$ $\beta = \rho/\varphi$ and $\gamma = \rho/(1 - v'(a))$ is exactly the left-hand side of (84), which implies that $da_t = 0$ along the optimal policy.

QED

Proof of Proposition 9

We apply Proposition V above. By Proposition 7, v is C^2 on $(-\infty, a_{max})$, strictly decreasing and strictly concave, and $\sup v < \infty$. The proof of Proposition 6 shows that (82) holds, the supremum over $\mathcal{U}^\varphi(K, \mathbb{P})$ being attained by the linear mechanism X^*. Proposition V gives the upper bound. The mechanism X^* has compensation rate $\varphi^* = \rho/\beta^*(a)$, $\kappa \equiv 1$, $\beta \equiv \rho/\varphi^*$ and $\beta^A \equiv 1$, and $a_t \equiv a$ by Proposition 8, so that (d) holds with $\varepsilon = a_{max} - a$. By Proposition 12, $\omega_t - \omega_0 = \big(g_{SB} - \frac{\beta^2\sigma^2 + \sigma_A^2}{2}\big)t + \beta\sigma Z_t + \sigma_A Z_t^A$, so that

$$\int e^{2\omega}\,\mathrm{d}\mathbb{P}_t = \exp\Big(2\big(g_{SB} - \frac{\sigma_A^2}{2}\big)t + \beta^2\sigma^2 t + 2\sigma_A Z_t^A\Big)\int e^{2\omega}\,\mathrm{d}\mathbb{P}.$$

Hence (c) holds along X^* if and only if $\int e^{2\omega}\,\mathrm{d}\mathbb{P} < \infty$; the other conditions in (c) hold because γ and φ^* are constant, $\omega - \log c_t(\omega) = a - \log\varphi^*$ and $A_t = e^a K_t$. Under this condition, $X^* \in \mathcal{A}^\varphi(K, \mathbb{P})$. Along X^*, the supremum in (82) is attained, $\theta_t = 0$ and $b_t = 1$, so that $r_t = \rho v(a)$ and $J^{(X^*)}(K, \mathbb{P}) = \log K + \mathbb{E}\int_0^\infty e^{-\rho t} r_t\,\mathrm{d}t = \log K + v(a)$.

Proof of Proposition 11

By Proposition 7, φ and γ maximize the Lagrangian

$$\log\gamma + \frac{1}{\rho}\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2}\Big) + \lambda\Big[\log\varphi + \frac{1}{\rho}\Big(\mu - \gamma - \varphi - \frac{\sigma_A^2}{2} - \frac{\sigma^2\rho^2}{2\varphi^2}\Big)\Big]$$

with $\lambda = -v'(a) > 0$. The incentive constraint binds, $\beta = \rho/\varphi$, so that $\sigma^2\beta^2 = \sigma^2\rho^2/\varphi^2$.
The first order condition with respect to γ yields

$$\gamma = \frac{\rho}{1+\lambda}.$$

The first order condition with respect to φ yields

$$-\frac{1}{\rho} + \lambda\Big(\frac{1}{\varphi} - \frac{1}{\rho} + \frac{\sigma^2\rho}{\varphi^3}\Big) = 0, \qquad \text{that is} \qquad \frac{\rho}{\varphi} + \frac{\sigma^2\rho^2}{\varphi^3} = \frac{1+\lambda}{\lambda}.$$

Combining the two yields

$$\frac{1}{1+\lambda} = 1 - \frac{\lambda}{1+\lambda} = 1 - \frac{\varphi^3}{\rho(\varphi^2+\rho\sigma^2)}, \qquad \text{hence} \qquad \gamma = \rho - \frac{\varphi^3}{\varphi^2+\rho\sigma^2}.$$

The growth rate of K is $\mu - \gamma - \varphi$, so that

$$g_{SB} = \mu - \gamma - \varphi = \mu - \rho - \frac{\rho\sigma^2\varphi}{\varphi^2+\rho\sigma^2},$$

and consequently (60). Finally, combining g_{SB} with Proposition 6 yields (59) and (61).

QED

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Online Appendix B: Background Theory

B.1. The Mean-field Limit

The purpose of this section is to describe the transition from an N-agent model to a mean-field model following the techniques developed in the book by Carmona and Delarue (2018). Next Proposition B1 shows that the economy with a continuum of agents is the limit of economies with N agents in which the principal uses the same feedback mechanism with linear compensation. In particular, a localized version of the optimal mechanism of Section 7 (see the end of this section), run in economies with N agents, generates state processes that converge to those of the continuum economy on every compact time interval.

More precisely, we fix a feedback mechanism with linear compensation that the principal uses in the economy with N agents and show that the resulting state processes converge, as $N \to \infty$, to the solution of the mean-field system (37)-(38), in which $\mathbb{P}_t$ is the conditional law of the continuation utility of an agent given the aggregate shocks. This result gives meaning to the economy with a continuum of agents, on which the principal's problem of Sections 6 and 7 is defined. We focus on logarithmic utility. Let $\mathcal{P}_e(\mathbb{R})$ be the set of probability measures $\mathbb{P}$ on $\mathbb{R}$ such that

$$\int e^{4|\omega|}\,\mathrm{d}\mathbb{P}(\omega) < \infty$$

and

$$\mathcal{E} = (0,\infty) \times \mathcal{P}_e(\mathbb{R}).$$

For $(K,\mathbb{P}) \in \mathcal{E}$, let $A = \int e^{\omega}\,\mathrm{d}\mathbb{P}(\omega)$ and $a = \log(A/K)$. We denote by $\mathcal{X}$ the set of feedback mechanisms with linear compensation of the form

$$\begin{gathered} c^P(K,\mathbb{P}) = \gamma(a)\,K, \qquad k(K,\mathbb{P},\omega) = K\,\frac{e^{\omega}}{A}, \qquad c(K,\mathbb{P},\omega) = \varphi(a)\,k(K,\mathbb{P},\omega), \\ \beta(K,\mathbb{P}) = \beta(a), \qquad \beta^A(K,\mathbb{P}) = \beta^A(a), \end{gathered} \tag{85}$$

such that

(H) the functions $\gamma, \varphi, \beta, \beta^A : \mathbb{R} \to \mathbb{R}$ are bounded and Lipschitz continuous, and there is $\varepsilon_0 > 0$ such that $\gamma \geq \varepsilon_0$ and $\varphi \geq \varepsilon_0$.

In $\mathcal{X}$, all agents are paid the compensation rate $\varphi(a)$, the capital allocation constraint $\int k\,\mathrm{d}\mathbb{P} = K$ holds by construction, and the incentive constraint $\beta \geq \rho k/c$ reads $\beta(a) \geq \rho/\varphi(a)$. The optimal mechanism of Section 7 has this form.

Let $Z^A, Z^1, Z^2, \ldots$ be independent Brownian motions, and let $\omega_0^1, \omega_0^2, \ldots$ be i.i.d. with law $\mathbb{P}$, independent of the Brownian motions. In the economy with N agents, the initial capital is K, the initial utilities are $\omega_0^1, \ldots, \omega_0^N$, and the principal uses a mechanism of $\mathcal{X}$ in which $\mathbb{P}_t$ is replaced by the empirical

measure $\mathbb{P}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\omega_t^{i,N}}$. We write $A_t^N = \frac{1}{N}\sum_i e^{\omega_t^{i,N}}$, $a_t^N = \log(A_t^N/K_t^N)$ and $\eta_t^i = e^{\omega_t^{i,N}} / \sum_j e^{\omega_t^{j,N}}$, so that

$$k_t^i = K_t^N \frac{e^{\omega_t^{i,N}}}{A_t^N} = N\,\eta_t^i K_t^N, \qquad c_t^i = \varphi(a_t^N)\,k_t^i, \qquad c_t^P = \gamma(a_t^N)K_t^N.$$

Then $\frac{1}{N}\sum_i k_t^i = K_t^N$, so that the capital allocation constraint (2) holds at every date, and $c_t^i = \varphi(a_t^N)k_t^i$ for every i, so that the compensation rate (6.1) holds at every date. If $\beta(a) \geq \rho/\varphi(a)$ for all a, the incentive constraint holds at every date for every agent. In the N-agent normalization of (9), the exposure of ω^i corresponds to $\beta^{ii} = N\beta$, and (25), $\beta^{ii} \geq \rho N k^i/c^i = \rho N/\varphi$, is equivalent to $\beta \geq \rho/\varphi$. The continuation utility of agent i is exposed to his own shock and to the aggregate shock only and the state equations (5) and (9) become, for $i = 1, \ldots, N$,

$$d\omega_t^{i,N} = \rho\big(\omega_t^{i,N} - \log c_t^i\big)dt + \sigma\beta(a_t^N)\,dZ_t^i + \sigma_A\beta^A(a_t^N)\,dZ_t^A, \tag{86}$$

$$dK_t^N = \Big(\mu K_t^N - \frac{1}{N}\sum_{i=1}^N c_t^i - c_t^P\Big)dt + \sigma_A K_t^N\,dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^N k_t^i\,dZ_t^i. \tag{87}$$

Since $\log c_t^i = \log\varphi(a_t^N) + \omega_t^{i,N} - a_t^N$ and $\frac{1}{N}\sum_i c_t^i = \varphi(a_t^N)K_t^N$, the system reads

$$d\omega_t^{i,N} = \rho\,(a_t^N - \log\varphi(a_t^N))\,dt + \sigma\beta(a_t^N)\,dZ_t^i + \sigma_A\beta^A(a_t^N)\,dZ_t^A, \tag{88}$$

$$\frac{dK_t^N}{K_t^N} = \big(\mu - \varphi(a_t^N) - \gamma(a_t^N)\big)dt + \sigma_A\,dZ_t^A + \sigma\sum_{i=1}^N \eta_t^i\,dZ_t^i. \tag{89}$$

The coefficients of the equation of $\omega^{i,N}$ depend on $\mathbb{W}^N = (\omega^{1,N}, \ldots, \omega^{N,N})$ only through a^N. Let $X^N = (\log K^N, \omega^{1,N}, \ldots, \omega^{N,N})$. The map $x \mapsto \log\frac{1}{N}\sum_j e^{x_j}$ has gradient $(\eta_i = \frac{e^{\omega^{i,N}}}{\sum_j e^{\omega^{j,N}}})_i$, whose components are nonnegative and sum to one, so that it is 1-Lipschitz for the sup norm, and a^N is a 1-Lipschitz function of X^N. The weights η_i have Jacobian $\eta^i(\delta_{ij} - \eta^j)$, so that each η^i and $H^N = \sum_i(\eta^i)^2$ are bounded Lipschitz functions of X^N. By (H), the coefficients of the system are globally Lipschitz, and this system has a unique strong solution.

Let Z be a Brownian motion and ω_0 a random variable with law $\mathbb{P}$, such that ω_0, Z and Z^A are independent, and let $(\mathcal{F}_t^A)$ be the filtration generated by Z^A. For the same linear mechanism of $\mathcal{X}$, let us define

$$K_t = K_0 \exp\Big(\int_0^t \big(\mu - \varphi(a_s) - \gamma(a_s) - \tfrac{1}{2}\sigma_A^2\big)ds + \sigma_A Z_t^A\Big), \tag{90}$$

$$\omega_t = \omega_0 + \rho\int_0^t (a_s - \log\varphi(a_s))\,ds + \sigma\int_0^t \beta(a_s)\,dZ_s + \sigma_A\int_0^t \beta^A(a_s)\,dZ_s^A, \tag{91}$$

and $\mathbb{P}_t = \mathcal{L}(\omega_t|\mathcal{F}_t^A)$.

Proposition B1 (Propagation of chaos). *Assume that (H) holds and that $\mathbb{P} \in \mathcal{P}_e$. Let $(K^N, \omega^{1,N}, \dots, \omega^{N,N})$ be the solution of (86)-(87), let a and K be given by (72) and (90), and, for $i \geq 1$, let $\bar{\omega}^i$ be defined by (91) with (ω_0^i, Z^i) in place of (ω_0, Z).*

(i) For every $T > 0$, there is a constant C_T, independent of N and i, such that

$$\mathbb{E}\Big[\sup_{t\leq T}\Big(|a_t^N - a_t|^2 + |\log K_t^N - \log K_t|^2 + |\omega_t^{i,N} - \bar{\omega}_t^i|^2\Big)\Big] \leq \frac{C_T}{N}.$$

(ii) Conditionally on $\mathcal{F}_\infty^A$, the processes $\bar{\omega}^1, \bar{\omega}^2, \dots$ are independent and identically distributed, and each of them solves (91). For every $m \geq 1$, $(K^N, \omega^{1,N}, \dots, \omega^{m,N})$ converges in probability, hence in law, to $(K, \bar{\omega}^1, \dots, \bar{\omega}^m)$ in $C([0,\infty); \mathbb{R}^{1+m})$ endowed with the topology of uniform convergence on compact sets.

(iii) For every $t \geq 0$ and every bounded Lipschitz function f,

$$\mathbb{E}\Big|\int f \,\mathrm{d}\mathbb{P}_t^N - \int f \,\mathrm{d}\mathbb{P}_t\Big| \leq \|f\|_{Lip}\sqrt{\frac{C_t}{N}} + \frac{\|f\|_\infty}{\sqrt{N}}.$$

For every T, the aggregate consumption of the agents $\varphi(a_t^N)K_t^N$, the consumption of the principal $\gamma(a_t^N)K_t^N$ and the ratio A_t^N/K_t^N converge, uniformly on $[0,T]$ and in probability, to $\varphi(a_t)K_t$, $\gamma(a_t)K_t$ and A_t/K_t.

Proof of Proposition B1. Hereafter, C_T denotes a constant that depends on T, on the Lipschitz constants of (H), on $\sigma, \sigma_A, \rho, \mu$ and on $\int e^{4|\omega|}\,\mathrm{d}\mathbb{P}_0(\omega)$, but not on N or i. It may change from line to line.
Let us recall the notations $\eta_t^i = e^{\omega_t^{i,N}} / \sum_j e^{\omega_t^{j,N}}$, $H_t^N = \sum_i (\eta_t^i)^2$ and $M_t^N = \int_0^t \sum_i \eta_s^i \, dZ_s^i$, so that $k_t^i/K_t^N = N\eta_t^i$ and $d\langle M^N\rangle_t = H_t^N\,dt$. Since the η_t^i are nonnegative and sum to one, $(\eta_t^i)^2 \leq \eta_t^i$ and, by the Cauchy–Schwarz inequality, $1 \leq NH_t^N$, so that $\frac{1}{N} \leq H_t^N \leq 1$. Itô's formula applied to (88) and (89) gives

$$d\log A_t^N = \Big(\rho\big(a_t^N - \log\varphi(a_t^N)\big) + \tfrac{1}{2}\sigma^2\beta(a_t^N)^2(1-H_t^N)\Big)dt + \sigma_A\beta^A(a_t^N)\,dZ_t^A + \sigma\beta(a_t^N)\,dM_t^N,$$
$$d\log K_t^N = \Big(\mu - \varphi(a_t^N) - \gamma(a_t^N) - \tfrac{1}{2}\sigma_A^2 - \tfrac{1}{2}\sigma^2 H_t^N\Big)dt + \sigma_A\,dZ_t^A + \sigma\,dM_t^N.$$

Subtracting the second equation from the first gives

$$da_t^N = \Big(F(a_t^N) + \tfrac{1}{2}\sigma^2\big(1-\beta(a_t^N)^2\big)H_t^N\Big)dt + \sigma_A\big(\beta^A(a_t^N)-1\big)dZ_t^A + \sigma\big(\beta(a_t^N)-1\big)dM_t^N, \tag{92}$$

where

$$F(a) = \rho\big(a - \log\varphi(a)\big) + \varphi(a) + \gamma(a) - \mu + \frac{\sigma^2\beta(a)^2 + \sigma_A^2}{2}.$$

The function F is the drift θ of (72), so that the continuum statistic satisfies $\mathrm{d}a_t = F(a_t)\,\mathrm{d}t + \sigma_A(\beta^A(a_t)-1)\,\mathrm{d}Z^A_t$. By (88), $\omega^{i,N}_t = Y^i_t + R^N_t$, where

$$Y^i_t = \omega^i_0 + \sigma\int_0^t \beta(a^N_s)\,dZ^i_s, \qquad R^N_t = \rho\int_0^t \big(a^N_s - \log\varphi(a^N_s)\big)ds + \sigma_A\int_0^t \beta^A(a^N_s)\,dZ^A_s.$$

The process R^N does not depend on i, so that $e^{R^N_t}$ cancels in $\eta^i_t = e^{Y^i_t}/\sum_j e^{Y^j_t}$. Hence

$$H^N_t = \frac{\sum_i e^{2Y^i_t}}{\big(\sum_j e^{Y^j_t}\big)^2} = \frac{1}{N}\,\frac{\frac{1}{N}\sum_i e^{2Y^i_t}}{\big(\frac{1}{N}\sum_j e^{Y^j_t}\big)^2}.$$

The function $x\mapsto x^{-2}$ is convex on $(0,\infty)$. Jensen's inequality applied to the empirical mean of $e^{Y^1_t},\dots,e^{Y^N_t}$ gives $\big(\frac{1}{N}\sum_j e^{x^j_t}\big)^{-2} \le \frac{1}{N}\sum_j e^{-2x^j_t}$, hence

$$H^N_t \le \frac{1}{N}\Big(\frac{1}{N}\sum_i e^{2Y^i_t}\Big)\Big(\frac{1}{N}\sum_j e^{-2Y^j_t}\Big). \tag{93}$$

Let $\bar\beta = \sup|\beta|$ and let $(\mathcal{F}_t)$ be the filtration generated by $(\omega^j_0)_{j\ge1}$, $(Z^j)_{j\ge1}$ and Z^A. The process Z^i is an $(\mathcal{F}_t)$-Brownian motion, ω^i_0 is $\mathcal{F}_0$-measurable and $\beta(a^N_s)$ is $(\mathcal{F}_s)$-adapted. For $\lambda=\pm4$, let

$$\mathcal{E}^\lambda_t = \exp\Big(\lambda\sigma\int_0^t \beta(a^N_s)\,dZ^i_s - \frac{\lambda^2\sigma^2}{2}\int_0^t \beta(a^N_s)^2\,ds\Big)$$

Since $|\beta|\le\bar\beta$, Novikov's condition holds, and $\mathcal{E}^\lambda$ is an $(\mathcal{F}_t)$-martingale with $\mathcal{E}^\lambda_0 = 1$. Since $8\sigma^2\int_0^t\beta(a^N_s)^2ds \le 8\sigma^2\bar\beta^2 t$,

$$e^{\lambda Y^i_t} = e^{\lambda\omega^i_0}\,\mathcal{E}^\lambda_t\,\exp\Big(8\sigma^2\int_0^t\beta(a^N_s)^2\,ds\Big) \le e^{8\sigma^2\bar\beta^2 t}\,e^{\lambda\omega^i_0}\,\mathcal{E}^\lambda_t.$$

Conditioning on $\mathcal{F}_0$ and using $\mathbb{E}[\mathcal{E}^\lambda_t|\mathcal{F}_0]=1$,

$$\mathbb{E}\big[e^{\lambda Y^i_t}\big] \le e^{8\sigma^2\bar\beta^2 t}\,\mathbb{E}\big[e^{\pm4\omega^i_0}\big] = e^{8\sigma^2\bar\beta^2 t}\int e^{\lambda\omega}\,\mathrm{d}\mathbb{P}_0(\omega). \tag{94}$$

Taking expectations in (93) gives

$$\mathbb{E}[H^N_t] \le \frac{e^{8\sigma^2\bar\beta^2 t}}{N}\Big(\int e^{4\omega}\,\mathrm{d}\mathbb{P}_0(\omega)\int e^{-4\omega}\,\mathrm{d}\mathbb{P}_0(\omega)\Big)^{1/2}.$$

Let $e_t = a^N_t - a_t$ and $\phi(t) = \mathbb{E}\big[\sup_{s\le t}|e_s|^2\big]$. By (H), $\log\varphi$ is Lipschitz continuous because φ and β^2 are Lipschitz continuous. Hence F, β and β^A are Lipschitz continuous, with a common constant L. Let $\bar\beta=\sup|\beta|$. We have,

$$\begin{aligned} e_t = e_0 &+ \int_0^t\big(F(a^N_s)-F(a_s)\big)ds + \frac{\sigma^2}{2}\int_0^t\big(1-\beta(a^N_s)^2\big)H^N_s\,ds \\ &+ \sigma_A\int_0^t\big(\beta^A(a^N_s)-\beta^A(a_s)\big)dZ^A_s + \sigma\int_0^t\big(\beta(a^N_s)-1\big)dM^N_s. \end{aligned}$$

The Cauchy–Schwarz inequality gives $\sup_{s\le t}\big|\int_0^s (F(a_u^N)-F(a_u))du\big|^2 \le TL^2\int_0^t \sup_{u\le s}|e_u|^2 ds$. Since $0\le H^N\le 1$, $\big(\int_0^t |1-\beta(a_s^N)^2|H_s^N ds\big)^2 \le (1+\bar\beta^2)^2 T\int_0^t H_s^N ds$. By Doob's inequality, the expected squared supremum of the integral in dZ^A is at most $4\sigma_A^2 L^2\int_0^t \phi(s)ds$. The integral in dM^N equals $\sum_i \int_0^t(\beta(a_s^N)-1)\eta_s^i\, dZ_s^i$, whose quadratic variation is $\int_0^t(\beta(a_s^N)-1)^2 H_s^N ds \le (\bar\beta+1)^2\int_0^t H_s^N ds$, so that, by Doob's inequality, its expected squared supremum is at most $4\sigma^2(\bar\beta+1)^2\int_0^t \mathbb{E}[H_s^N]ds$. Hence, for $t\le T$,

$$\phi(t) \le C_T\Big(\mathbb{E}|e_0|^2 + \int_0^t \phi(s)\,ds + \int_0^T \mathbb{E}[H_s^N]\,ds\Big) \le C_T\Big(\frac{1}{N} + \int_0^t \phi(s)\,ds\Big).$$

Because F is Lipschitz continuous, the other coefficients of (92) are bounded and $\mathbb{E}|a_0^N|^2<\infty$. Grönwall's lemma gives $\phi(T)\le C_T/N$. By (90),

$$\log K_t^N - \log K_t = -\int_0^t \big(\varphi(a_s^N)-\varphi(a_s)+\gamma(a_s^N)-\gamma(a_s)\big)ds - \frac{\sigma^2}{2}\int_0^t H_s^N\,ds + \sigma M_t^N.$$

By (H), $0\le H^N\le 1$, Doob's inequality $\mathbb{E}\sup_{s\le t}|M_s^N|^2 \le 4\int_0^t \mathbb{E}[H_s^N]ds$ and thus,

$$\mathbb{E}\Big[\sup_{t\le T}|\log K_t^N - \log K_t|^2\Big] \le C_T\Big(\phi(T) + \int_0^T \mathbb{E}[H_s^N]\,ds\Big) \le \frac{C_T}{N}.$$

By (88) and (91),

$$\begin{aligned}\omega_t^{i,N} - \bar\omega_t^i &= \rho\int_0^t \Big(a_s^N - a_s - \log\varphi(a_s^N) + \log\varphi(a_s)\Big)ds \\ &\quad + \sigma\int_0^t \big(\beta(a_s^N)-\beta(a_s)\big)dZ_s^i + \sigma_A\int_0^t \big(\beta^A(a_s^N)-\beta^A(a_s)\big)dZ_s^A.\end{aligned}$$

Every integrand is bounded by a constant times $|e_s|$, so that, by Doob's inequality, $\mathbb{E}\big[\sup_{t\le T}|\omega_t^{i,N}-\bar\omega_t^i|^2\big] \le C_T\phi(T) \le C_T/N$. This proves (i).
The process a is a functional of Z^A. Hence $\bar\omega^i = \Phi(\omega_0^i, Z^i, Z^A)$ for a measurable map Φ that does not depend on i. The pairs (ω_0^i, Z^i) are i.i.d. and independent of Z^A. Therefore the $\bar\omega^i$ are i.i.d. conditionally on $\mathcal{F}_\infty^A$, and $\mathcal{L}(\bar\omega_t^i|\mathcal{F}_t^A) = \mathcal{L}(\bar\omega_t^i|\mathcal{F}_\infty^A)$ does not depend on i. Recall that the topology of uniform convergence on compact sets is given by the distance $d(x,y) = \sum_{n\ge1} 2^{-n}\min\big(1, \sup_{t\le n}|x_t-y_t|\big)$. We have shown that $\sup_{t\le n}|\omega_t^{i,N}-\bar\omega_t^i|$ and $\sup_{t\le n}|\log K_t^N - \log K_t|$ converge to 0 in probability for every n and every $i\le m$. Since $\sup_{t\le n} K_t<\infty$ almost surely, $\sup_{t\le n}|K_t^N - K_t|$ also converges to 0 in probability. Each term of the series defining d is bounded by 2^{-n}, and dominated convergence gives

$$\mathbb{E}\big[d\big((K^N,\omega^{1,N},\dots,\omega^{m,N}),(K,\bar\omega^1,\dots,\bar\omega^m)\big)\big] \to 0.$$

Finally, let f be bounded and Lipschitz continuous. Then

$$\Big|\int f\,\mathrm{d}\mathbb{P}_t^N - \int f\,\mathrm{d}\mathbb{P}_t\Big| \le \frac{\|f\|_{Lip}}{N}\sum_{i=1}^N |\omega_t^{i,N}-\bar\omega_t^i| + \Big|\frac{1}{N}\sum_{i=1}^N f(\bar\omega_t^i) - \int f\,\mathrm{d}\mathbb{P}_t\Big|.$$

The expectation of the first term is at most $\|f\|_{Lip}(C_t/N)^{1/2}$. Conditionally on $\mathcal{F}^A_\infty$, the variables $f(\bar\omega^i_t)$ are i.i.d. with mean $\int f\,\mathrm{d}\mathbb{P}_t$, so that the conditional second moment of the second term is at most $\|f\|^2_\infty/N$, and its expectation is at most $\|f\|_\infty/\sqrt{N}$. Finally, the aggregate consumption of the agents is $\frac{1}{N}\sum_i c^i_t = \varphi(a^N_t)K^N_t$, the consumption of the principal is $\gamma(a^N_t)K^N_t$ and $A^N_t/K^N_t = e^{a^N_t}$. Since φ and γ are bounded and Lipschitz continuous, $\sup_{t\le T}|a^N_t - a_t|$ and $\sup_{t\le T}|K^N_t - K_t|$ converge to 0 in probability, and $\sup_{t\le T}(|a_t| + K_t) < \infty$ almost surely, these quantities converge uniformly on $[0,T]$ in probability to $\varphi(a_t)K_t$, $\gamma(a_t)K_t$ and $e^{a_t} = A_t/K_t$.

QED

The optimal mechanism of Section 7, $(\gamma^*, \varphi^*, \beta^*, \beta^{A^*})$ with $\beta^* = \rho/\varphi^*$ and $\beta^{A^*} = 1$, has the form (85) but does not satisfy (H): it is defined on $(-\infty, a_{max})$ only, $\varphi^*(a) \to 0$ and $\beta^*(a) \to \infty$ as $a \to -\infty$, and $\gamma^*(a) \to 0$ as $a \to a_{max}$. Let $J = [\alpha, \alpha'] \subset (-\infty, a_{max})$ with $\alpha < a_0 < \alpha'$, let π_J be the projection of $\mathbb{R}$ onto J, and let

$$\gamma_J = \gamma^* \circ \pi_J, \qquad \varphi_J = \varphi^* \circ \pi_J, \qquad \beta_J = \rho/\varphi_J, \qquad \beta_{A,J} = 1.$$

By Proposition 7, γ^* and φ^* are positive and continuously differentiable on $(-\infty, a_{max})$, so that this localized mechanism satisfies (H). Its drift function F_J coincides with F^* on J, and $F^*(a_0) = 0$ by Proposition 8. Hence the solution of (72) is $a_t \equiv a_0$, and the continuum economy under the localized mechanism is the continuum economy under the optimal mechanism.

Proposition B1 is a statement on state processes over compact time intervals, and its constants C_T grow with T. It does not imply the convergence of discounted infinite-horizon payoffs, nor the approximate optimality of the continuum solution in economies with finitely many agents. Nor does it imply that, for fixed N, the process $\omega^{i,N}$ defined by (86) is the continuation utility generated by the consumption c^i, which requires $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}[\omega^{i,N}_T] = 0$.

B.2. Mean-field Control

In a discounted infinite horizon mean-field control problem (also known as a McKean–Vlasov control problem), an optimizer (here the principal) seeks to maximize a functional over an infinite time horizon, where the dynamics depend not only on the state and control but also on the statistical distribution of the entire population. The shift from classical control to mean-field control is essentially a shift to the case in which the state variable is a probability distribution. In mean-field control, the principal looks at the distribution $\mathbb{P}$ of the continuation utilities of the agents and asks what is the optimal utility from this distribution. The principal's value function is $V : \mathcal{E} \to [-\infty, +\infty)$, with $V(K, \mathbb{P}) \le \log K + v(a)$ on $\{a < a_{max}\}$ by Proposition 9,

$$V(K, \mathbb{P}) = \sup_X \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c^P_t) dt\right]. \tag{95}$$

The state space is $\mathcal{E}$ and not $(0,\infty) \times \mathcal{P}_2(\mathbb{R})$, where $\mathcal{P}_2(\mathbb{R})$ is the Wasserstein space of measures with finite second moment. The aggregate $A = \int e^\omega\,\mathrm{d}\mathbb{P}(\omega)$,

on which the value function depends, may be infinite for $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$, and it is not continuous for the Wasserstein distance. Condition (c) of Appendix A.3 ensures that the martingale terms in Lemma 13 below are square integrable and that its drift terms are integrable. The set $\mathcal{P}_e(\mathbb{R})$ is not a vector space, but it is convex: if $\mu, \nu \in \mathcal{P}_e(\mathbb{R})$ and $\varepsilon \in [0,1]$, then $(1-\varepsilon)\mu + \varepsilon\nu \in \mathcal{P}_e(\mathbb{R})$. Convexity is all that is needed to define derivatives with respect to the measure as directional derivatives along mixtures, as in Carmona and Delarue (2018), Vol. 1, Chapter 5. We do not need a topology on $\mathcal{E}$, because the verification argument only uses Itô's formula along the flow $t \mapsto (K_t, \mathbb{P}_t)$. In this section, admissible mechanisms are those of the set $\mathcal{A}^{\varphi}(K, \mathbb{P})$ defined in Appendix A.3. By Lemma 13 below, A_t is finite for all t, so that $(K_t, \mathbb{P}_t) \in \mathcal{E}$ for all t, almost surely.

Let $\mathcal{G}$ be the set of continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that $|g(\omega)| \leq C(1 + e^{\omega})$ for some constant C. Every $g \in \mathcal{G}$ is integrable with respect to every $\mathbb{P} \in \mathcal{P}_e(\mathbb{R})$.

Definition. *Let $F : \mathcal{P}_e(\mathbb{R}) \to \mathbb{R}$ and $\mu \in \mathcal{P}_e(\mathbb{R})$.*

- *F admits a first variation at μ if there exists a function $\nabla F[\mu] \in \mathcal{G}$ such that, for all $\nu \in \mathcal{P}_e(\mathbb{R})$,*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(F((1-\varepsilon)\mu + \varepsilon\nu) - F(\mu)\Big) = \int_{\mathbb{R}} \nabla F[\mu](\omega)\, \mathrm{d}(\nu - \mu)(\omega).$$

  The function $\nabla F[\mu]$ is defined up to an additive constant.

- *F admits a second variation at μ if, for every ω, the function $\mu \mapsto \nabla F[\mu](\omega)$ admits a first variation at μ, denoted $\nabla^2 F[\mu](\omega, \cdot)$.*

- *F is regular at μ if $\nabla F[\mu]$ is twice continuously differentiable and $\nabla^2 F[\mu]$ admits a continuous mixed derivative $\partial^2_{\omega\omega'} \nabla^2 F[\mu](\omega, \omega')$. We denote by $\partial_\omega \nabla F[\mu]$ and $\partial_{\omega\omega} \nabla F[\mu]$ the first and second derivatives of $\nabla F[\mu]$*[22]*.*

Let X be an admissible mechanism and $(K_t, \mathbb{P}_t)$ the associated state, where $\mathbb{P}_t$ is the conditional law of ω_t given $\mathcal{F}^A_t$. Write $b(\omega) = \rho(\omega - \log c(K_t, \mathbb{P}_t, \omega))$, $\beta(\omega) = \beta(K_t, \mathbb{P}_t, \omega)$ and $\beta^A(\omega) = \beta^A(K_t, \mathbb{P}_t, \omega)$. For a regular function F,

[22] When F is also defined on $\mathcal{P}_2(\mathbb{R})$ and satisfies the continuity and growth conditions of Carmona and Delarue (2018), Vol. 1, Chapter 5, $\partial_\omega \nabla F[\mu]$ coincides with the L-derivative. We do not use this identification. The functionals of Section 7 are not continuous for the Wasserstein distance.

Itô's formula along $t \mapsto \mathbb{P}_t$ reads

$$\begin{aligned} dF(\mathbb{P}_t) = &\Big(\int_{\mathbb{R}} \partial_\omega \nabla F[\mathbb{P}_t](\omega)\, b(\omega)\, \mathbb{P}_t(d\omega)\Big)dt \\ &+ \frac{1}{2}\Big(\int_{\mathbb{R}} \big(\sigma^2\beta(\omega)^2 + \sigma_A^2\beta^A(\omega)^2\big)\partial_{\omega\omega} \nabla F[\mathbb{P}_t](\omega)\, \mathbb{P}_t(d\omega)\Big)dt \\ &+ \frac{\sigma_A^2}{2}\Big(\iint \beta^A(\omega)\beta^A(\omega')\, \partial^2_{\omega\omega'} \nabla^2 F[\mathbb{P}_t](\omega,\omega')\, \mathbb{P}_t(d\omega)\mathbb{P}_t(d\omega')\Big)dt \\ &+ \Big(\int_{\mathbb{R}} \partial_\omega \nabla F[\mathbb{P}_t](\omega)\, \sigma_A\beta^A(\omega)\, \mathbb{P}_t(d\omega)\Big) dZ_t^A. \end{aligned} \tag{96}$$

When $\sigma_A = 0$, the last two terms vanish and $\mathbb{P}_t$ is the law of ω_t. Carmona and Delarue (2018) prove (96) on $\mathcal{P}_2(\mathbb{R})$ (Vol. 1, Chapter 5, Th. 5.92, without aggregate shock, and Vol. 2, Chapter 4, Th. 4.14, with aggregate shock), under growth conditions on the derivatives that exclude the exponential growth of the functions we use. We only apply (96) to the class of functions below, for which we prove it directly in Lemma 13. By applying the dynamic programming principle, this Itô formula allows for the derivation of the HJB equation (41) for the value function V. Let $\psi : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with $|\psi'(\omega)| + |\psi''(\omega)| \le Ce^\omega$, so that $|\psi(\omega)| \le C'(1 + e^\omega)$, and let h be twice continuously differentiable on an open set $D \subset (0,\infty) \times \mathbb{R}$. We consider the function F defined, for $(K, \int \psi \,\mathrm{d}\mu) \in D$, by

$$F(K,\mu) = h\Big(K, \int_{\mathbb{R}} \psi(\omega)\, \mu(d\omega)\Big).$$

Let $y = \int \psi \,\mathrm{d}\mu$. For fixed K, $F(K,\cdot)$ is regular at every $\mu \in \mathcal{P}_e(\mathbb{R})$, with

$$\begin{aligned} \nabla F[\mu](\omega) &= h_y(K,y)\,\psi(\omega), & \nabla^2 F[\mu](\omega,\omega') &= h_{yy}(K,y)\,\psi(\omega)\psi(\omega'), \\ \partial_\omega \nabla F[\mu] &= h_y(K,y)\,\psi', & \partial^2_{\omega\omega'} \nabla^2 F[\mu](\omega,\omega') &= h_{yy}(K,y)\,\psi'(\omega)\psi'(\omega'), \\ \partial_{\omega\omega} \nabla F[\mu] &= h_y(K,y)\,\psi''. \end{aligned}$$

The value function of Section 7 belongs to this class, with $\psi(\omega) = e^\omega$ and $h(K,y) = \log K + v(\log y - \log K)$.

Here h is defined on $D = \{(K,y) : K > 0,\ 0 < y < Ke^{a_{max}}\}$, and condition (d) of Appendix A.3 keeps (K_t, A_t) in D.

Lemma 13. *Let $X \in \mathcal{A}^\varphi(K,\mathbb{P})$, let ψ be as above, let $Y_t = \int \psi \,\mathrm{d}\mathbb{P}_t$, and write $b(\omega) = \rho(\omega - \log c_t(\omega))$. Then Y is a continuous semimartingale and*

$$dY_t = \Big(\int \Big(\psi' b + \frac{1}{2}\psi''\big(\sigma^2\beta^2 + \sigma_A^2\beta^{A\,2}\big)\Big)\mathrm{d}\mathbb{P}_t\Big)dt + \sigma_A\Big(\int \psi'\beta^A \,\mathrm{d}\mathbb{P}_t\Big)dZ_t^A.$$

In particular, $d\langle K,Y\rangle_t = \sigma_A^2 K_t\big(\int \psi'\beta^A \,\mathrm{d}\mathbb{P}_t\big)dt$ and $d\langle Y\rangle_t = \sigma_A^2\big(\int \psi'\beta^A \,\mathrm{d}\mathbb{P}_t\big)^2 dt$. If $(K_t,Y_t) \in D$ for all t, then $F(K_t,\mathbb{P}_t) = h(K_t,Y_t)$ satisfies Itô's formula for the pair (K_t,Y_t). Written with the derivatives of F listed above, this formula is (96) completed by the terms $\partial_K F\, dK_t + \frac{1}{2}\partial_{KK}F\,\sigma_A^2 K_t^2\, dt$ and by the cross-variation term $\big(\int \partial^2_{\omega K}\nabla F[\mathbb{P}_t](\omega)\,\sigma_A^2\beta^A(\omega)K_t\,\mathbb{P}_t(d\omega)\big)dt$, which are the terms of (41).

Proof. Itô's formula applied to $\psi(\omega_t)$ gives

$$\psi(\omega_t) = \psi(\omega_0) + \int_0^t \Big(\psi' b + \frac{1}{2}\psi''\big(\sigma^2\beta^2 + \sigma_A^2 {\beta^A}^2\big)\Big)(\omega_s)\, ds + \sigma \int_0^t \psi'(\omega_s)\beta(\omega_s)\, dZ_s + \sigma_A \int_0^t \psi'(\omega_s)\beta^A(\omega_s)\, dZ_s^A.$$

Since $|\psi(\omega)| \leq C'(1 + e^\omega)$, $\psi(\omega_0)$ is integrable. Since $|\psi'| + |\psi''| \leq Ce^\omega$, condition (c) of Appendix A.3 implies that the ds-integrand is integrable on $[0,t] \times \Omega$ and that both stochastic integrals are square-integrable martingales. Let $(\mathcal{F}_s)$ be the filtration generated by ω_0, Z and Z^A. We take conditional expectations given $\mathcal{F}_t^A$. Let ξ be square integrable and $\mathcal{F}_t^A$-measurable. By the martingale representation theorem in the filtration of Z^A, $\xi = \mathbb{E}[\xi] + \int_0^t \eta_s\, dZ_s^A$ for an $(\mathcal{F}_s^A)$-progressive process η. For an $(\mathcal{F}_s)$-progressive process H with $\mathbb{E}\int_0^t H_s^2\, ds < \infty$, this gives $\mathbb{E}[\xi \int_0^t H_s\, dZ_s] = 0$, since $\langle Z, Z^A\rangle = 0$, and $\mathbb{E}[\xi \int_0^t H_s\, dZ_s^A] = \mathbb{E}\int_0^t H_s\eta_s\, ds = \mathbb{E}[\xi \int_0^t \mathbb{E}[H_s|\mathcal{F}_s^A]\, dZ_s^A]$. Hence the conditional expectation of the dZ integral is zero, and that of the dZ^A integral is $\sigma_A \int_0^t \mathbb{E}[\psi'(\omega_s)\beta^A(\omega_s)|\mathcal{F}_s^A]\, dZ_s^A$. Since the increments of Z^A after s are independent of $\mathcal{F}_s$, $\mathbb{E}[g_s|\mathcal{F}_t^A] = \mathbb{E}[g_s|\mathcal{F}_s^A]$ for $s \leq t$ and every integrable $\mathcal{F}_s$-measurable g_s, so that the conditional expectation of the ds integral is $\int_0^t \mathbb{E}[\,\cdot\,|\mathcal{F}_s^A]\, ds$ (see also Carmona and Delarue (2018), Vol. 2, Chapter 4). The controls at date s are functions of $(K_s, \mathbb{P}_s, \omega_s)$, where $(K_s, \mathbb{P}_s)$ is $\mathcal{F}_s^A$-measurable and $\mathbb{P}_s$ is the conditional law of ω_s given $\mathcal{F}_s^A$, so that $\mathbb{E}[g(K_s, \mathbb{P}_s, \omega_s)|\mathcal{F}_s^A] = \int g(K_s, \mathbb{P}_s, \omega)\, \mathrm{d}\mathbb{P}_s(\omega)$. This gives the dynamics of Y. The brackets follow from $dK_t = ((\mu - \varphi_t)K_t - c_t^P)\, dt + \sigma_A K_t\, dZ_t^A$. Itô's formula for $h(K_t, Y_t)$ holds because h is C^2 on the open set D and the continuous semimartingale (K_t, Y_t) stays in D. Since $\partial_\omega \nabla F = h_y\psi'$, $\partial_{\omega\omega}\nabla F = h_y\psi''$, $\partial^2_{\omega\omega'}\nabla^2 F = h_{yy}\psi'(\omega)\psi'(\omega')$ and $\partial^2_{\omega K}\nabla F = h_{Ky}\psi'$, the terms $h_y\, dY_t$, $\frac{1}{2}h_{yy}\, d\langle Y\rangle_t$ and $h_{Ky}\, d\langle K, Y\rangle_t$ are the corresponding terms of (96) and of (41).

QED