Rational Bubbles in Realistic Markets?
An OLGA Model *

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Abstract

The classical models of rational bubbles are based on intergenerational trading frictions. Here, we devise a model of bubbles based on overlapping generations of assets (OLGA) rather than of agents (OLG). Rational dynastic agents face idiosyncratic income risk and are subject to borrowing-constraints. Stochastically emerging Lucas trees are tradable in a stock market, but only after their emergence. In a numerical example, we find that bubbles may exist for realistic degrees of stock market incompleteness. The model offers an alternative explanation for stock price anomalies that are usually ascribed to investor irrationality.

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1 Introduction

If people have rational expectations, can asset prices systematically exceed assets’ fundamental value? According to Samuelson (1958) and Tirole (1985), the answer is affirmative. Within the OLG model developed by Allais (1947), they show that there are conditions under which the demand for stores of value creates room for such “rational bubbles”. However, in an influential theoretical contribution Santos and Woodford (1997) argue that these conditions are unlikely to be fulfilled in practice, because intergenerational trading frictions cease to matter if a significant fraction of households behave dynastically, as in Barro (1974). Then, under seemingly weak assumptions, asset supply is sufficiently large to ensure that the demand for consumption-smoothing will not support asset price bubbles. Santos and Woodford (1997, page 48) summarize their insights as follows:

These results suggest that known examples of pricing bubbles depend upon rather special circumstances. In consequence, familiar examples (such as the overlapping generations model of Samuelson (1958) or the Bewley (1980) model) seem to be quite fragile as potential foundations for monetary theory. For when these models are extended to allow for trading in additional assets that are sufficiently productive (in particular, capital earning returns satisfying the criterion of Abel et al.), or to include an infinitely lived household (or Barro “dynasty”) that is able to borrow against its future endowment and that owns a fraction of aggregate wealth, pricing bubbles – and hence monetary equilibria – can be excluded under quite general assumptions.

It is uncontroversial that many models of rational bubbles rely on unrealistically incomplete asset markets. Yet, if Santos and Woodford are right, it follows that the risk-free real interest rate $r$ must be above the rate of growth $g$. This is not what the data say. And if $r < g$ for some securities, the next question is inevitable: Is it pos-

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1 Other notable early contributions to this literature include Wallace (1980), Blanchard and Watson (1982), and Weil (1987); for further references, see Blanchard and Fischer (1989, Chapter 5).

2 For an example of how the view of Santos and Woodford has become conventional wisdom, see LeRoy (2004, page 801), where he reluctantly concludes: “Within the neoclassical paradigm there is no obvious way to derail the chain of reasoning that excludes bubbles”.

3 During 1930-2011, US GDP growth was on average 3.29%; average per capita growth was 2.11%. Average real return to holding government debt was 0.41% on 3 months T-bills and 1.94% on 10 year bonds. Recent projections from the Congressional Budget Office likewise indicate that that government debt of average maturity will yield a return below the rate of growth many years into the future. The longer bonds are not risk-free.
sible that the assumptions of Santos and Woodford are also too strong, but produce unrealistically complete securities markets?

To circumvent the contentious issue of intergenerational trading frictions, our analysis considers a dynastic endowment economy of the kind developed by Huggett (1993), which in turn builds on Bewley (1980, 1983). It is well known that such dynastic heterogeneous-agents economies can also in principle possess bubbles due to demand for consumption-smoothing. But as Santos and Woodford indicate, these bubbles are often thought to be artifacts of unrealistically sparse securities markets. For example, Bewley does not admit any durable asset except money – which is then a rational bubble. Huggett (1993) does admit a limited amount of (real) borrowing, but such personal debts constitute the only security in which wealthy individuals can save. Severe scarcity of securities is also a feature of more recent work that seeks to explain low risk-free interest rates with the help of heterogeneous agent models, such as Krusell, Mukoyama, and Smith (2011).

Our contribution is to demonstrate that bubbles can exist in economies of this kind even if we admit a realistically well-functioning stock market. There can be bubbles on stocks, and the model is also consistent with observed stock price anomalies such as the twin-shares puzzle (Rosenthal and Young, 1990; Mitchell, Pulvino, and Stafford, 2002). In the twin-shares puzzle, most famously associated with Royal Dutch and Shell, two securities with identical dividend streams trade at markedly different prices. Hitherto, this puzzle and others like it have been interpreted as proof of investor irrationality and motivate a sizable fraction of the behavioral finance literature (Barberis and Thaler, 2003; Scheinkman, 2014).

Since our results differ from those of Santos and Woodford (1997), we are obliged to identify which of their assumptions we discard. As the above quote indicates, Santos and Woodford provide two sets of sufficient conditions for the non-existence of bubbles. One set of sufficient conditions involve borrowing constraints, effectively saying that rational bubbles cannot exist if people can borrow fully against all their future

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4In principle, the analysis might as well have been conducted in an economy with production, such as the model of Aiyagari (1994). However, the additional complexity blurs the relationship to Santos and Woodford (1997).

5A difference with the classical OLG models is that savings here are used to smooth random fluctuations in income or tastes rather than predictable variation in income.

6We do not here wish to claim that rational bubble theory offers a more plausible explanation for such puzzles than does irrational bubble theory. Indeed, rational bubble theory does not offer an explanation for the bubbles that have sometimes been observed in finite horizon securities markets. However, the empirical regularity that short-sale frictions are associated with pricing anomalies does not reliably discriminate between rational and irrational bubble theories. Some limit to arbitrage is crucial to both. Section 5 discusses how the borrowing constraint in our model prevents the agents from arbitraging away a rational bubble.
non-traded incomes (their “endowments”). It is already broadly acknowledged that this condition is often violated in practice, and it is routinely violated in asset pricing models too.7 Accordingly, our analysis will assume that people without current liquid assets cannot borrow at all (but this condition can be relaxed considerably).

Santos and Woodford’s second set of sufficient conditions does not involve borrowing constraints, but instead requires that securities markets are sufficiently deep to “span” the non-traded incomes. While this requirement may appear superficially plausible, Santos and Woodford actually assume that all securities that are traded in the future are descendants of securities that are traded today. Roughly speaking, at time $t$ each available security yields a payout in the form of consumption goods (or money) and new securities to be traded at time $t + 1$. There are no completely new securities at time $t + 1$. This assumption is directly responsible for many of Santos and Woodford’s (1997) results – for example, the assumption quite immediately implies that bubbles “can never start:” If there is an equilibrium bubble at time $t$, there must also have been an equilibrium bubble at time $t − 1$. By contrast, we assume that at each time $t$ agents know that new assets will emerge in the future, yet the dividends associated with these assets cannot be securitized at date $t$.8 The steady inflow of new superstar entrepreneurs illustrate the relevance of our alternative assumption. As a case in point, consider Facebook, an internet company started by novices in 2003. When it became publicly traded in 2010, the company was worth around 40 billion US dollars. Under the assumptions of Santos and Woodford (1997), well diversified investors would already have held financial claims on Facebook and any other venture that the founder Mark Zuckerberg may have initiated, long before 2003.

Our analysis focuses solely on the classical consumption-smoothing rationale for bubbles. There is also a growing complementary literature on rational bubbles which links saving and investment, focusing on the demand for stores of value by borrowing-constrained firms rather than borrowing-constrained households. Farhi and Tirole (2012) is a good port of entry to this literature; see also Woodford (1990), Kocherlakota (2011), Giglio and Severo (2012), Martin and Ventura (2012), Wang and Wen (2012), and a series of papers by Miao and Wang; especially Miao and Wang (2012, 2015a, 2015b).9

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7 In particular, there is by now a large literature on endogenous borrowing constraints arising from limited commitment. For central theoretical contributions in settings directly related to bubble theory, see for example, Scheinkman and Weiss (1986), Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jerman (2000), and Hellwig and Lorenzoni (2009).

8 Similar securitization frictions are extensively but informally discussed in connection to rational bubbles already by Tirole (1985). For a particularly close parallel to our OLGA formulation, albeit not in the context of bubbles, see Gârleanu, Kogan, and Panageas (2012).

9 Since we ignore production, we abstract from investment. We also ignore short-term trading frictions and thus securities’ properties as media of exchange; we thus doubly neglect the work of Kiyotaki...
This literature has remained silent with respect to Santos and Woodford’s objection that, in reality, large and liquid stock markets should allow agents to arbitrage away the bubbles. Our analysis suggests that this silence has been warranted. Realistic stock markets do not preclude rational bubbles.

The analysis proceeds as follows. Section 2 provides a simple stochastic dynamic general equilibrium model of an endowment economy without growth, allowing a clean comparison with Santos and Woodford (1997). Section 3 demonstrates that the degree of securitization implied by Santos and Woodford’s assumptions is unrealistically high; our key parameter $\sigma$ must equal 1, implying that no new assets appear and current securities represent claims on all assets that will ever exist. Section 4 contains a numerical example, which indicates that realistic levels of securitization are consistent with the existence of bubbles. Section 5 contests the view that unrestricted short-selling ought to eliminate asset price bubbles. The Appendix discusses positive productivity growth. In this case, bubbles can exist alongside infinitely-lived productive assets, just as in Tirole’s (1985) OLG model with positive population growth.

2 The Model

Time is discrete, and the horizon is infinite. Period $t = 0$ refers to the current period. Periods $t = -\infty, ..., -1$ comprise the history and determine the “initial conditions” that characterize period 0. Periods $t = 1, ..., \infty$ comprise the future. There is a continuum of infinitely lived agents distributed along the unit interval. $^{10}$

Preferences: Agents consume a homogeneous good and have identical preferences. Their utility function is

$$U = E \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $c_t$ is consumption in period $t$, and $\beta \in (0, 1)$ is the subjective discount factor.

Technology: There are two kinds of productive assets. First, there is a continuum of length $A$ of productive and tradable Lucas trees. In each period that a tree is alive, it yields an amount $d \geq 0$ of non-storable fruit. Survival is an i.i.d. process; each tree survives to the next period with probability $\sigma < 1$. Correspondingly, a measure $(1-\sigma)A$ trees die each period. Likewise, a measure $(1-\sigma)A$ trees are born each period. $^{11}$ Of the trees that emerged $v$ years ago, there thus remain $(1-\sigma)^v A$. Each

$^{10}$As usual, each infinitely lived agent could be seen as representing an altruistically linked dynasty.

$^{11}$In this sense, our model replaces the assumption of overlapping generations of agents (OLG) with overlapping generations of assets (OLGA).
new tree is paired with a random agent, who initially has full ownership of this tree. For convenience, we assume that an agent can absorb at most one new tree in any period. Thus, in each period a fraction \((1 - \sigma)A\) of the agents receive a new tree.

Second, there is a continuum of length 1 of infinitely lived non-tradable trees. Each agent owns one non-tradable tree; to distinguish them from the tradable trees, we refer to them as bushes from now on. In each period, each agent recieves \(\epsilon^y_t\) units of fruit from their bush. The amount \(\epsilon^y_t\) can take one of a finite number of values in the set \(Y\) and evolves through time according to a first-order Markov chain with transition probabilities defined by the matrix \(\Pi\).

There also exists a continuum of length \(S\) of an unproductive asset that has no intrinsic value but which is perfectly durable and can be traded. We call this asset stamps.

Let \(\epsilon^a_t\) be an indicator variable, taking the value 1 if an agent gets one of the new trees in period \(t\) and 0 otherwise. Let \(\epsilon_t = (\epsilon^y_t, \epsilon^a_t)\) and let \(\epsilon^t = \{\epsilon_0, ..., \epsilon_t\}\) denote the partial sequence from period 0 up to period \(t\) and let \(\mathcal{E}^t\), denote the corresponding set of all possible such sequences. Define probability measures \(\gamma^t(x_0, \cdot) : \mathcal{E}^t \to [0, 1]\), \(t = 0, 1, ...\) where, for example, \(\gamma^t(x_0, \epsilon^t)\) is the probability of history \(\epsilon^t\) given an agent’s initial state \(x_0\) specified below.

Trade: At any time, agents can trade fruit, stamps and stocks in existing trees in a frictionless market. In principle, they are also able to borrow against collateral (shares in trees), but not against the income from bushes. As will become clear, this assumption can be relaxed somewhat, but an occasionally tight borrowing constraint is required to support bubbles. However, our main focus is on another crucial financial friction.

**Assumption 1.** Claims on trees that have not yet emerged cannot be traded.

Prices: As we seek to endogenize the value of financial assets, the consumption good, fruit, is our numeraire.

For simplicity, we assume that all units of the unproductive asset are indistinguishable. Thus, they must be priced the same. The price of one unit of the unproductive asset at time \(t\) is denoted \(p^s_t\).

While the productive assets are also intrinsically identical, in the sense that each tree yields the same expected future return, their age differs. We thus allow the price to differ across generations of trees, letting \(p^j_{t,t}\) denote the price of a tree of generation \(j \leq t\) at time \(t\). As will become clear, we could as well have let the price depend on other labels that the firms have. A key point of our analysis is to show that assets that have identical dividends can nonetheless have different prices, and age is just a
particularly simple illustration.

Agents’ wealth: Let \( s_t \) denote the quantity of unproductive asset that an agent possesses at the end of period \( t - 1 \). Let \( a_{j,t} \) denote the quantity of generation \( j \) assets that an agent possesses at the end of period \( t - 1 \), and let
\[
a_t = \{a_{j,t}\}_{j=-\infty}^{t-1}
\]
denote an agent’s holdings of all productive assets born in previous periods. Let \( x_0 = (a_0, s_0, \varepsilon_0) \in X \) denote the initial state of an agent, where \( X = \mathbb{R}^\infty \times \mathbb{R} \times Y \times \{0, 1\} \). Let \( \mathcal{X} = \mathcal{R}^\infty \times \mathcal{R} \times Y \times \{0, 1\} \) where \( \mathcal{R} \) and \( Y \) denote the Borel sets that are subsets of \( \mathbb{R} \) and \( Y \), respectively. (The set \( \mathcal{X} \) includes the set of all possible relevant states for the economy.) Observe that we admit negative asset holdings. Thus, short-selling is allowed, as long as the agent does not violate an overall borrowing constraint to be specified below.

Behavior: In period 0, given its initial state \( x_0 \), each agent chooses consumption and savings for each possible sequence \( \varepsilon^t \). Let \( \phi_t : \mathcal{E}^t \to X_{as}, t = 0, 1, \ldots \) describe the savings plan, where \( X_{as} = \mathbb{R}^\infty \times \mathbb{R} \) and \( \phi_{a, j,t}(\varepsilon^t; x_0) \) denotes the value for \( a_{j,t+1} \) that is chosen in period \( t \) if the history up to \( t \) is \( \varepsilon^t \), conditional on the agent’s initial state being \( x_0 \). Similarly \( \phi_{s,t}(\varepsilon^t; x_0) \) denotes the value for \( s_{t+1} \). For all \( t \), the savings plan must satisfy a no-borrowing constraint
\[
\sum_{j=-\infty}^{t} p_{j,t} a_{j,t+1} + p_t^s s_{t+1} \geq 0,
\]
(2) where \( a_{j,t+1} = \phi_{a,j,t}(\varepsilon^t; x_0) \) and \( s_{t+1} = \phi_{s,t}(\varepsilon^t; x_0) \). Let \( c_t : \mathcal{E}^t \to \mathbb{R}_+ \) describe the associated plan for consumption.

At the beginning of period \( t \), a fraction \( 1 - \sigma \) of the trees alive at \( t - 1 \) will have died; the remaining will yield a dividend \( d \). An agent’s budget constraint is therefore given by
\[
c_t(\varepsilon^t; x_0) = \varepsilon_t^\gamma + p_t^s s_t + \sum_{j=-\infty}^{t-1} (p_{j,t} + d) \sigma a_{j,t} + (p_{t,t} + d) \varepsilon_t^a - \sum_{j=-\infty}^{t} p_{j,t} a_{j,t+1} - p_t^s s_{t+1}.
\]
(3)
The agent’s problem is thus to maximize expected discounted lifetime utility
\[
\sum_{t=0}^{\infty} \sum_{\varepsilon^t \in \mathcal{E}^t} \beta^t u \left( c_t(\varepsilon^t; x_0) \right) \gamma^t(x_0, \varepsilon^t)
\]
(4) through a set of choices \( c_t(\varepsilon^t; x_0) \) and \( \phi_t(\varepsilon^t; x_0) \) for all \( t \), subject to (2), (3).
\[ c_t(\epsilon^t; x_0) \in \mathbb{R}_+, \]
\[ \phi_t(\epsilon^t; x_0) \in X_{\text{abs}}, \]

and taking as given the sequences of prices

\[ \left\{ \{ p_{j,t} \}^{t-1}_{j=-\infty}, p^s_t \right\}_{t=0}^{\infty} \]

and the initial state \( x_0 \).

**Markets and distribution:** The distribution of agents over the initial state is described by a measure \( \kappa : X \rightarrow [0,1] \). By integrating over \( \kappa \), aggregate variables can be computed. Market clearing in financial markets implies that

\[ S = \int_X s_0 d\kappa, \tag{5} \]
\[ \sigma^{-j}(1 - \sigma) A = \int_X a_{j,0} d\kappa, \quad \forall \ j = -\infty, \ldots, -1, \tag{6} \]

and for all \( t \geq 0 \)

\[ S = \int_X \sum_{\epsilon^t \in \mathcal{E}^t} \phi_{s,t}(\epsilon^t; x_0) \gamma_t(x_0, \epsilon^t) d\kappa \tag{7} \]

and

\[ \sigma^{-j}(1 - \sigma) A = \int_X \sum_{\epsilon^t \in \mathcal{E}^t} \phi_{a,j,t}(\epsilon^t; x_0) \gamma_t(x_0, \epsilon^t) d\kappa, \quad \forall \ j = -\infty, \ldots, t. \tag{8} \]

Similarly, market clearing in the market for fruit implies that

\[ d + y = \int_X \sum_{\epsilon^t \in \mathcal{E}^t} c_t(\epsilon^t; x_0) \gamma_t(x_0, \epsilon^t) d\kappa, \quad \forall \ t \geq 0, \]

where

\[ y = \int_X \sum_{\epsilon^t \in \mathcal{E}^t} \epsilon^t y \gamma_t(x_0, \epsilon^t) d\kappa, \quad \forall \ t \geq 0 \]

is the total amount of fruit all bushes yield in each period (since we ignore growth, these left-hand side variables do not need an index \( t \)).

**Equilibrium:** An equilibrium comprises sequences of prices \( \left\{ \{ p_{j,t} \}^{t-1}_{j=-\infty}, p^s_t \right\}_{t=0}^{\infty} \) and sequences of decisions \( \left\{ c_t(\epsilon^t; x_0), \phi_t(\epsilon^t; x_0) \right\}_{t=0}^{\infty} \) for all \( x_0 \in X \) and \( \epsilon^t \in \mathcal{E}^t \), together with probability measures \( \left\{ \gamma_t(x_0, z) \right\}_{t=0}^{\infty} \) for all \( x_0 \in X \) and \( z \in \mathcal{E}^t \), and a measure \( \kappa(x) \) for all \( x \in X \) describing the initial distribution such that

1. the decision rules solve the agents’ problem given prices and the initial state \( x_0 \);  
2. all markets clear, and
3. the measure $\gamma^t(x_0, \epsilon^t)$ satisfies

(a) $\gamma^t\left(x_0, \left\{\epsilon^{t-2}, \{\epsilon^y_i, z\}, \{\epsilon^y_j, 0\}\right\}\right) = \sigma \Pi_{ij} \gamma^{t-1}\left(x_0, \left\{\epsilon^{t-2}, \{\epsilon^y_i, z\}\right\}\right)$ for all $\epsilon^y_i, \epsilon^y_j \in y$ and $z \in \{0, 1\}$,

(b) $\gamma^t\left(x_0, \{\epsilon^{t-2}, \{\epsilon^y_i, z\}, \{\epsilon^y_j, 1\}\right\}) = (1 - \sigma) \Pi_{ij} \gamma^{t-1}\left(x_0, \{\epsilon^{t-2}, \{\epsilon^y_i, z\}\right\})$ for all $\epsilon^y_i, \epsilon^y_j \in y$ and $z \in \{0, 1\}$, and

(c) $\gamma^0(x_0, z) = 1$ if $\epsilon_0 \in z \in \mathcal{E}^0$ and 0 otherwise.

3 Analysis

We focus on stationary equilibria, where returns to saving and the aggregate distribution of wealth are constant, both with respect to agents and asset classes.

Let $R^{a}_{j,t} = \sigma(p_{j,t+1} + d)/p_{j,t}$ denote the return to holding a tree of generation $j$ and $R^{s}_t = p_{t+1}^s/p_t^s$ denote the return to holding stamps. Since there is no aggregate uncertainty and all existing trees give fruit in any equilibrium, no-arbitrage implies that $R^{a}_{j,t} = R^{s}_t$. Moreover, if stamps are valued no-arbitrage implies that $R^{s}_t = R^{a}_{t}$. Stationary equilibria have some time-invariant equilibrium rate of return $R$ (which generally differs across equilibria). Since $R = \sigma(p + d)/p > \sigma$, the fundamental value of a tree in a stationary equilibrium is given by

$$p_f(R) = \frac{\sigma d}{R} + \frac{\sigma^2 d}{R^2} + \ldots = \frac{\sigma d}{R - \sigma}.$$ (9)

Since stamps do not yield real dividends, their fundamental value is zero.

Stationary equilibria potentially come in four types. We say that the equilibrium has no bubble when $p^s = 0$ and $p_j = p_f(R)$ for all $j$; that it has a pure stamp-bubble when $p^s > 0$ and $p_j = p_f(R)$ for all $j$; that it has a pure tree-bubble when $p^s = 0$, and $p_j > p_f(R)$ for some $j$; and that it has a mixed bubble if both $p^s > 0$ and $p_j > p_f(R)$ for some $j$. The conditions required for these different equilibria to exists are best understood by considering three versions of our economy.

Economy 1: $A = 0$. There are no productive trees. This is the economy studied by Huggett (1993). It is well known that in this economy there exist both a no-bubble equilibrium and a bubble equilibrium. In the former, agents live in autarchy (the borrowing constraint is zero) and consume the endowment from their bush. In the latter, the return $R = 1$, the price of stamps is $p^s > 0$ and the aggregate bubble is simply $p^s S$. 
Economy 2: $\sigma = 1$. In this version, immortal (infinitely lived) productive trees are added to the Huggett economy. Since the return to trees is $R = (p + d)/p > 1$, a bubble cannot exist. If it did it would outgrow the economy, as noted by Scheinkman (1980) in a related setting. Initially, this observation was taken to imply that bubbles are unlikely, because they would be dominated by infinitely-lived assets such as land. In reality, and thus in a suitable extension of the model, there are two reasons why productive land does not destroy the possibility of rational bubbles. First, as pointed out by Tirole (1985), in economies with positive growth bubbles can coincide with infinitely-lived productive assets as long as their dividends grow more slowly than the economy as a whole. (Of course, the equilibrium is then no longer stationary, except in the limit when the share of output from land approaches zero.) This argument holds equally well in our OLGA setting as in Tirole’s OLG setting, and the Appendix shows how to introduce productivity growth in the model. (Indeed, when dividend growth is slower on immortal trees than on than on mortal trees, there may even be bubbles on the immortal trees.) Second, even if productivity growth on land is the same as on other assets, bubbles may still exist on stocks (mortal trees) in a stationary equilibrium if there is a positive land tax or if there are transaction costs associated with trading land. Thus, the central issue is to understand the conditions under which there can be bubbles on mortal trees. We now turn to this issue.

Economy 3: $\sigma < 1$. In this version, productive trees with uncertain longevity are added to the Huggett economy. The return to these mortal trees is $R = \sigma(p + d)/p$, which may or may not be smaller than 1 depending on the parameters of the model. This implies that the bubble equilibrium may exists when $\sigma < 1$. But this is not to say that $\sigma = 1$ is a knife-edge condition for bubbles. In the numerical example below, $\sigma$ must be non-trivially below 1 before stationary bubbles emerge.

If the bubble equilibrium exists, it comes in two forms: with a pure stamp-bubble (where the aggregate bubble is $p^{\text{S}}$) or with a pure tree-bubble. Pure-tree bubble equilibria come in many different shapes, but we focus on equilibria in which the bubble is the same on all trees in a generation and each new generation is born with the same bubble. Let $p$ be the price of each tree in the most recent generation. By definition, the bubble in the price of a new tree is $(p - p_f(R))$. Over time the bubble in the price an individual tree grows at rate $(R/\sigma) - 1 > 0$. But since trees die at rate $(1 - \sigma)$ the total bubble on a generation of trees falls at rate $R - 1 < 0$. Since new trees keep emerging the aggregate bubble is constant and equal to $(1 - \sigma)A (p - p_f(R)) / (1 - R)$. Moreover, the oldest existing tree carries the largest bubble in the economy, but the youngest generation of trees carries the largest share of the aggregate bubble on trees.\footnote{The latter property is a consequence of the class of equilibria that we focus on and not a general feature of all rational bubble equilibria.}
In this version of the model, which has zero growth, there cannot be a bubble on trees if there is also a bubble on stamps. The reason is that a bubble on stamps requires \( R = 1 \). Thus, the bubble on each generation of trees must be constant. But since new generations of trees keep emerging, the aggregate bubble must be growing, a contradiction. However, this feature vanishes when the economy grows (see the Appendix).

Note that the interest rate cannot be too low if bubbles are too exist. In the pure-tree equilibria considered here, this is easiest seen by noting that if the price of a new tree carries no bubble, then the aggregate bubble is zero. This occurs when the price of a new tree equals its fundamental value. Since the fundamental value is decreasing in the interest rate, there is a threshold interest rate, \( R_0 \), at which \( p = p_f(R) \). Thus pure-tree bubble equilibrium requires an interest rate in the interval \( 1 > R > R_0 \geq \sigma \). We discuss this further in the numerical example below.

**Relationship with Santos and Woodford (1997):** The economy studied here is closely related to the environment investigated by Santos and Woodford (1997) (henceforth SW). They consider an economy with two income processes. One process is attached to individual agents; SW calls this the “endowment of consumption goods” (they denote it \( \omega \)). This endowment is inalienable and thus corresponds to income from bushes, \( \varepsilon y \).

The other income process (they denote it \( z \)) is attached to “securities.” Securities represent state-dependent claims to consumption goods as well as to new securities, and they are alienable. Translated to our model, SW-securities correspond to claims on all productive trees, now and in the future. More precisely, as future trees emerge, they are immediately securitized, and the new securities are paid out as part of the dividend to the holders of already existing securities. A central assumption undergirding SW’s main results is that, for any future state, it is possible to purchase securities today whose real dividends (directly and indirectly through the dividends of offspring securities) are greater than the income from inalienable assets (“the aggregate endowment”) in that state; this is their condition (2.9). If it satisfied, bubbles cannot exist.

Translated to our model, SW’s condition (2.9) is satisfied as long as \( \sigma^t dA \geq y \). Clearly, our model violates SW’s condition (2.9) when \( \sigma < 1 \). The point is simple: In a world where new assets arrive at a strictly positive rate (and these new assets are not all offspring of old assets), old assets will eventually become unimportant.\(^{13}\)

Of course, (2.9) is only a sufficient condition for the non-existence of bubbles in Santos and Woodford’s model. We need to also check directly that there is no other

\(^{13}\)Note that it is the emergence of new assets rather than the death of old assets that is crucial for the argument. This distinction becomes relevant when growth is positive.
profitable portfolio trading strategy. In particular, we must consider the possibility of short-selling assets with a bubble. We postpone that analysis until Section 5.

The importance of Assumption 1: Suppose, counterfactually, that claims on trees that have not yet emerged can be traded, i.e., Assumption 1 does not hold. No-arbitrage conditions then imply identical returns to a diversified portfolio of existing trees and to one of non-existing trees. Let \( p_{t}^{t+k} \) denote the price in period \( t \) on a diversified portfolio of claims on the \((1−\sigma)A\) trees that will emerge in period \( t+k \). The price of such a portfolio is simply the present discounted value of the claims it contains, that is

\[
p_{t}^{t+k} = \frac{(1−\sigma)(p+d)}{R^k}.
\]

Since trade in claims on all future non-existing trees is allowed in this case (i.e, \( k = 1, ..., \infty \)), in equilibrium the return is \( R > 1 \); otherwise the price of claims on trees that emerge far into the future would be infinite. Hence bubbles are ruled out if such claims can be traded.

4 Numerical Example

We next solve the model numerically. For simplicity, we now assume that the dividend from bushes is constant. If claims on trees that have not yet emerged could be traded, we would thus be in a complete markets economy; there would be no consumption uncertainty for any agent. The equilibrium distribution of wealth would be degenerate, and the rate of return would be given by \( 1/\beta \). The equilibrium price of trees would be \( p_{j,t} = \frac{\sigma d}{1/\beta−\sigma} \) for all \( j, t \).

4.1 Parametrization

Normalize \( S = A = 1 \). Let a period be one year, set the utility discount factor \( \beta \) to 0.97. Let the felicity function \( u \) take the CRRA form

\[
u(c) = \frac{c^{1−\mu}}{1−\mu},
\]

and set the coefficient of relative risk aversion \( \mu \) to 3.\(^{14}\) The yield from trees is \( d = 1 \), and the yield from bushes is \( y = 0.1 \). Since bushes can not be traded we think of

\(^{14}\)Kimball, Sahm, and Shapiro (2009) provide some of the most recent estimates. Using survey results in the PSID they find that the average \( \mu \) is 4.19 (see their Table 1). Since our main results are generally strengthened as \( \mu \) increases, we prefer to be conservative.
its yield as the level of safety nets in the economy. With average consumption equal to $d + y$, the poorest agent in the economy, one with zero assets, can thus consume approximately 9 percent of average consumption.\footnote{As a comparison, having an income of 9 percent of average income in the US would place an individual in approximately the bottom 5th percentile (see Díaz–Giménez, Glover and Ríos-Rull, 2011). Also, as of 2010, the maximum monthly allotment of food stamps for one person in the US is $200 per month. According to the Bureau of Labor Statistics, average annual expenditure for single individual households was $30,613 in 2011, which implies that food stamps provide a safety net of slightly less than 8 percent of average consumption.}

Let us finally consider relevant values for the key parameter, $\sigma$. As mentioned in the Introduction, we think of $1 - \sigma$ as a measure of new securities (IPOs) relative to the stock of existing securities. Rather than choosing a single value, we solve the model for all yearly survival rates $\sigma \in [0.85, 1]$. Any reasonable estimate is surely in this range.\footnote{Jovanovic and Rosseau (2005) find, using US data between 1885 and 2003, that the value of newly listed firms has on average been just above 3.1 percent of total stock market value, with highs of above 10 percent in periods of rapid technological change such as the electrification era and the IT era. In the period in between these eras, the value of newly listed firms gradually increase; between 1930-1979 the average value was 1.4 percent of total stock market value and between 1980-2003 it was 3.6 percent. As the criteria for being listed on the stock exchanges are quite stringent, these numbers can be thought of as a loose lower bound for new firm creation. According to Caves (1997), who considers a broad sample of firms in eight different countries during the 1970s and early 1980s (using data assembled by John Cable and Joachim Schwalbach), average annual entry rates are 7.7 percent in the US and as high as 13 percent in Belgium. Average exit rates are very similar to entry rates. Using more recent data covering 24 countries, Bartelsman et al. (2004) find that gross turnover (exit plus entry rates) are between 20-25 percent across all firms. Among large firms, defined as those with at least 20 employees, exit and entry rates are around five percent. In the time series dimension, they find that between 15-20 percent of firms in industrial countries fail during the first two years, that across all countries 65 percent of firms survive the first four years while only 30-50 percent of all firms survive beyond seven years. This points to an average yearly rate of destruction/creation of 8 percent.}

### 4.2 Solution algorithm

The solution to the agents’ problem is computed using Carroll’s (2006) endogenous grid method. Since the returns to stamps and a perfectly diversified portfolio of trees are equal in any equilibrium with valued stamps, agents are indifferent between holding trees and stamps in such equilibria. For simplicity we thus assume that all agents hold the same portfolio shares. We then compute the stationary distribution by approximating the invariant density function.

### 4.3 Equilibria

Figure 1 displays the equilibrium price of the most recent generation of trees, $p$, as a function of $\sigma$, in three different classes of equilibria; (i) no-bubbles (ii) pure stamp-bubbles (iii) pure tree-bubbles. As explained below, the model does not admit mixed
bubbles in equilibrium. The figure also displays the equilibrium price of trees if markets are complete.

![Diagram](image)

**Figure 1: Asset prices as functions of the survival rate $\sigma$**

With complete markets, the price of trees increase as their survival rate increases, since the discounted stream of dividends increases.

With incomplete markets but no bubbles, the price of trees first increases and then falls as the survival rate is increased. The non-monotonicity reflects two competing effects. On the one hand, as the survival rate increases trees become more valuable since the expected dividends are greater. On the other hand, the demand for precautionary saving falls.$^{17}$

A pure stamp-bubble equilibrium exists for all survival rates $\sigma < 0.9799$. With a bubble on stamps, the price of trees monotonically increases as the survival rate in-

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$^{17}$The strength of the precautionary motive depends on the relative size of the bush and and the dividends from trees; the larger the bush, the smaller is the precautionary motive. For a large enough bush, the non-monotonicity disappears.
creases, just as under complete markets. This is natural; in both cases we are discounting a growing stream of dividends at a constant interest rate ($R = 1$ in the concentrated bubble equilibrium and $R = 1/\beta$ in the complete markets economy).

For all survival rates $\sigma < 0.9799$ there also exist multiple pure tree-bubble equilibria. The shaded area in Figure 1 shows the price of new trees in these equilibria. For example, consider the various tree-bubble equilibria that lie on the vertical line between points $A$ (the no-bubble equilibrium) and $B$ (the pure stamp-bubble equilibrium) in the figure. In $A$ the price of trees is high, implying a low real interest and thus a high fundamental value of a new tree, $\sigma d / (R - \sigma)$. More specifically, in $A$ the interest is equal to threshold interest rate $R$. This low real interest implies low aggregate saving, and in $A$ the high fundamental value fully absorbs the low aggregate saving. That is, in $A$ the real interest rate, $R$, is so low that it is inconsistent with bubbles.

A downward movement from $A$ to $B$ is associated with a lower price on new trees but a higher real interest rate. This implies that the fundamental value of a new tree monotonically falls between $A$ and $B$. The bubble value of a new tree however changes non-monotonically. In $A$ there are no bubbles, and in $B$ the bubble is concentrated on stamps. Hence, the bubble on new trees first increases and then decreases as we move from $A$ to $B$. But even if the bubble on each new tree is smaller in the equilibria close to $B$, the real interest rate is higher, and the bubble on a generation of trees thus falls at a slower rate. Therefore, the aggregate bubble increases monotonically as we move from $A$ to $B$ and is largest in $B$ where stamps are the only bubble.

In terms of ex-ante average welfare these pure tree-bubble equilibria can easily be ranked; the larger the aggregate bubble the higher is average welfare. The larger are the bubbles, the higher is the return to saving, and the smaller is the consumption inequality. The best equilibrium from an ex ante perspective therefore occurs in the limit where the bubble is concentrated on stamps.

When the economy has a pure tree-bubble of the kind we consider here, the prices of older trees have bigger bubbles than those of younger trees. However, the general insight is that price bubbles on assets with identical fundamental values can be of different sizes, not that age is correlated with bubbliness. For example, it is easy to construct equilibria in which only some generations of trees have bubbles. Likewise, there might be bubbles on a strict subset of the trees in a generation and not on the remainder.
5 Short-selling

Why is it that different prices for assets with identical dividend streams can occur in a rational-expectations equilibrium despite our assumption that short-selling of the more expensive asset is allowed? For example, consider the simple strategy of short-selling the more expensive asset, using one part of the revenue buy the cheaper asset in order to cover dividend payments on the shorted asset, and consuming the remainder, corresponding to the value of the bubble – say, $b$. If the cheaper asset would suffice as collateral for the shorted asset, this strategy allows an arbitrage profit. However, the cheaper asset does not provide enough collateral to cover the market value of the shorted asset. At time $t$, the net debt associated with the shorted asset equals $n(t) = b(1 + r)^t$. Thus, in order to respect the borrowing constraint, the agent must always hold other assets that are worth at least this much; once the value of other assets drops below $n(t)$ the agent needs to close the position.\[^{18}\] In our model with uninsurable income shocks, agents will eventually always be up against the borrowing constraint at some time in the future (otherwise, their consumption plan would not be optimal). Thus, the agent’s welfare is exactly the same as it would have been if the extra consumption at date 0 had been funded by selling assets valued at $b$ (the same assets that have now served as collateral instead). The scheme of infinite horizon arbitrage fails.\[^{19}\]

6 Conclusion

The Nobel lectures of Eugene Fama and Robert Shiller illustrate two prevalent views among asset pricing specialists. Shiller (2014) argues that irrational bubbles are common and have great macroeconomic impact. Fama (2014) argues that there is little or no evidence of irrational bubbles on such broad aggregates as the stock price index. In particular, Fama argues that it is very difficult to spot any of the arbitrage opportunities that irrational bubbles imply. Neither Fama nor Shiller mention the possibility

\[^{18}\text{Again, we do not here address the reasons why the borrowing constraints exist in the first place, but refer to the rich literature on financial frictions in general (see footnote}[7]\text{ and limits to arbitrage in particular; see Gromb and Vayanos (2010) for a survey. While these arguments were developed as a foundation for the theory of irrational bubbles (Shleifer and Vishny, 1997), they apply with similar force to our theory of rational bubbles.}\]

\[^{19}\text{Borrowing constraints limit short-selling to an even greater extent in reality. For example, the Federal Reserve Board (under Regulation T) requires all short sale accounts to have 150 percent of the value of the short sale at the time the sale is initiated, and while subsequent maintenance margins can be lower, collateral requirements typically exceed 125 percent of the shorted asset’s market value. Furthermore, there is typically a positive interest rate on the borrowing of securities that are sold short, a “haircut.”}\]
of rational bubbles, a theory that permits much of the mispricing that concerns Shiller while at the same time predicting the lack of arbitrage opportunity that concerns Fama.

What is the reason for this neglect? The theory of rational bubbles has a distinguished pedigree; it was largely developed by four earlier Nobel laureates – Maurice Allais, Paul Samuelson, Peter Diamond, and Jean Tirole – and in the late 1980’s the theory used to constitute a core chapter in the graduate macroeconomics curriculum (Chapter 5, simply called ”Bubbles”, takes up more than one hundred pages in the textbook by Blanchard and Fischer, 1989). However, around that time two influential criticisms of rational bubble theory started to take hold, and most mainstream economists came to embrace either the position that bubbles are due to irrationality or that bubbles do not exist.20

One of the two influential criticisms was theoretical and is epitomized by Santos and Woodford (1997). It said that the key assumptions required for rational bubbles are too strong. The present paper argues that this criticism is less convincing than it first appears.

The second influential criticism of rational bubble theory was empirical. Abel et al (1989) argued that investments are generally smaller than the returns to investment. However, as they themselves pointed out, their conclusion hinges on the assumption that those returns are really returns to reproducible capital and not to non-reproducible factors such as land or entrepreneurship. More recent contributions by Caselli and Feyrer (2007) and Geerolf (2013) cast doubt on that assumption. However, since the present model considers an endowment economy, it must be silent on this empirical issue. A natural next step is to pursue a more extensive quantitative investigation that takes seriously the presence of reproducible capital. Preliminary results indicate that that there can be sizable rational bubbles in a calibrated macroeconomic model with productivity growth, reproducible capital, and public debt.

Another natural extension is to model crash risk. It has been understood at least since Blanchard and Watson (1982) that rational bubbles might also be stochastic; they may burst with positive probability. If such bursting is correlated, it could be a source of aggregate risk. A model with this form of aggregate risk would provide a channel from asset price bubbles and the asset price puzzles that are related to risk premiums – the excess volatility puzzle, the value premium puzzle, and the equity premium.

20There are also other objections to rational bubble theory, but most are based on simple misconceptions (some of which we have been guilty of ourselves). For example, it is easy to think that there cannot be bubbles on stocks because the average return to owning stocks is above the economy’s growth rate. This criticism apparently forgets both that dividends reduce the fundamental value component and that fundamental values and bubble values need not grow at the same rate. As long as a sufficient fraction of the stock return compensates for non-insurable risk, high stock returns are compatible with bubbles.
References


Appendix: Positive Growth

For brevity, we merely mention the main differences with the analysis above. Assume the yields from trees and bushes grows at a constant exogenous rate $g > 0$, thus $d_t = (1 + g)d_t$ and $y_t = (1 + g)y_{t-1}$. Moreover assume that in each period $t$ an amount $g_t^s S_t$ new stamps appear, so that $S_{t+1} = (1 + g_t^s)S_t$; the value of $g_t^s$ will be determined in equilibrium. New stamps are randomly paired with agents. We assume that the probability of receiving new stamps is independent of productivity and of the probability of receiving a new tree. Let $\varepsilon_{i,t}^s S_t$ denote the number of new stamps agent $i$ is endowed with; for simplicity we assume that agents are either endowed with new stamps ($\varepsilon_{i,t}^s = 1$) or are not endowed ($\varepsilon_{i,t}^s = 0$). The probability of being endowed with new stamps is thus $g_t^s$. An agent’s budget constraint is then

$$c_t(\varepsilon^t; x_0) = \varepsilon^y_t + p_t^s (s_t + \varepsilon_t^s S_t) + \sum_{j=-\infty}^{t-1} (p_{j,t} + d) \sigma a_{j,t} + (p_{t,t} + d) \varepsilon_t^d - \sum_{j=-\infty}^{t} p_{j,t} a_{j,t+1} - p_t^s s_{t+1},$$

where $\varepsilon_t = (\varepsilon_t^y, \varepsilon_t^d, \varepsilon_t^s)$, $x_0 = (a_0, s_0, \varepsilon_0) \in X$ and $X = R^\infty \times R \times Y \times \{0,1\}^2$. Let $\Gamma$ denote the Markov transition matrix that describes how $\varepsilon_t$ evolves over time.

The agent’s problem is thus to choose a feasible plan $\{c_t(\varepsilon^t; x_0), \phi_t(\varepsilon^t; x_0)\}_{t=0}^\infty$ to maximize expected discounted lifetime utility (4) subject to (2), (11), the transition matrix $\Gamma$, anticipated sequences of prices, dividends, taxes and transfers, and the initial state $x_0$.

Since the economy features exogenous growth of productivity and endogenous growth in stamps it is convenient to normalize variables by their respective growth rate. Let a tilde denote that the variable is de-trended. Most variables are de-trended by productivity, for example $\tilde{\epsilon}_t \equiv \tilde{\epsilon}_t / \lambda_t$. The exceptions are the following; $\tilde{p}_t^s \equiv p_t^s S_t / \lambda_t$, and $\tilde{s}_t \equiv \tilde{s}_t / \lambda_t$. Note that the de-trending implies that we normalize aggregate supply of stamps to one in all periods; $\tilde{S}_t = 1$ for all $t$.

Equilibrium: An equilibrium comprises sequences of prices $\{\{\tilde{p}_{j,t}\}_{j=-\infty}^{t-1}, \tilde{p}_t^s\}_{t=0}^\infty$, growth rates of stamps $\{g_t^s\}$, and sequences of decisions $\{\tilde{c}_t(\varepsilon^t; x_0), \tilde{\phi}_t(\varepsilon^t; x_0)\}_{t=0}^\infty$ for all $x_0 \in X$ and $\varepsilon^t \in \mathcal{E}^t$, together with probability measures $\{\gamma^t(x_0, z)\}_{t=0}^\infty$ for all $x_0 \in X$ and $z \in \mathcal{E}^t$, and a measure $\kappa(x)$ for all $x \in \mathcal{X}$ describing the initial distribution such that (i) the decision rules solve the agents’ problem given prices and the initial state $x_0$, (ii) all markets clear, and (iii) the measure $\gamma^t(x_0, \varepsilon^t)$ is consistent with the transition matrix $\Gamma$.

Analysis: As before, stationary equilibria can be characterized via the no-arbitrage conditions. Let the variables $\tilde{R}_{j,t} = \sigma(\tilde{p}_{j,t+1} + \tilde{d}_{t+1}) / \tilde{p}_{j,t}$, and $\tilde{R}_t^s = \tilde{p}_{t+1}^s / [\tilde{p}_t^s (1 + g_t^s)]$.
denote the growth-adjusted returns to holding trees and stamps respectively, that is \( \tilde{R}^{a}_{j,t} = R^{a}_{j,t}/(1 + g) \) and \( \tilde{R}^{s}_{t} = R^{s}_{t}/(1 + g) \). The fundamental value of a tree is now
\[
p_f(\tilde{R}) = \sigma\tilde{d}\tilde{R} - \sigma.
\]
In any equilibrium, no-arbitrage implies that \( \tilde{R}^{a}_{j} = \tilde{R} \) for all \( j \). As without growth, price bubbles on trees can exist if \( 1 > \tilde{R} > \tilde{R} \geq \sigma \), where the lower threshold \( \tilde{R} \) is such that price of a new tree is \( \tilde{p} = p_f(\tilde{R}) \). That is, if the real interest is below the real growth rate \( (R < 1 + g) \), but not so low that the fundamental assets values in the economy fully absorb aggregate saving \( (R > R \geq (1 + g)\sigma) \).

If stamps are valued then no arbitrage implies that \( 1 \geq \tilde{R}^{s} = \tilde{R} > \tilde{R} \) where \( R^{s} = (1 + g)/(1 + g^{s}) \). Thus for stamps to be valued, the real interest rate must be below or equal to the real growth rate. Moreover, if stamps are valued, the growth rate of stamps is endogenously determined so that \( g^{s} \geq g \). That is, the aggregate value of unproductive assets like stamps can only remain a significant fraction of the economy’s overall asset value if there is steady creation of new assets at least as large as the growth rate of the economy. This was first noted by Tirole (1985).

Suppose now that there also exists real assets such as land in this economy. To the extent that land lives forever, we can analyze this case by simply setting \( \sigma = 1 \). The key condition for bubbles to exist on land, is that the growth rate of the real dividend from land, \( g_{l} \), must be smaller than the economy’s real rate of interest, \( R - 1 \), which is smaller than the economy’s average rate of growth \( g \). To the extent that productivity grows faster for other input factors (capital, labor, ideas) than for land, this condition is reasonable.