Market for Manipulable Information

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Abstract

We study how investors, firms, and information sellers interact in a market with manipulable information. Our model builds on the framework of Admati and Pfleiderer (1986) and introduces two new features: information manipulability and investor heterogeneity. In the baseline model where investors care about actual characteristics, the average degree of signal manipulability has no effect on the equilibrium, whereas the uncertainty about signal manipulability plays a key role. Its contribution depends on firms’ incentive to manipulate the signals that are used to generate the score, which in turn depends on the equilibrium price sensitivity to the score. The optimal design of the score in this setting weights the precision of different signals against the endogenous uncertainty from manipulation. The introduction of mandate investors, who care about the scores on the characteristics and not the characteristics themselves, generates a new incentive for information sellers to inflate the scores. Pushing too strongly on the mandate could lead to reduction in the informativeness of the score and the equilibrium price, and could even result in mandate investors holding less of the desired stocks.

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1 Introduction

Financial markets are in an age of information explosion. An area that has witnessed particularly striking growth is the so-called “alternative data” (examples include social media data, satellite or aerial imagery, and geolocation data), which are widely used in forecasting company performances and stock returns, as well as for certification (such as credit ratings or ESG ratings). Compared to “standard data” (such as financial and accounting, government statistics), which typically are subject to well-established reporting and auditing requirements, alternative data can be more timely, but also more prone to manipulation. The goal of this paper is to study how investors, firms, and information sellers interact in a market with manipulable information.

Our model builds on the framework of Admati and Pfleiderer (1986) and introduces two new features: information manipulability and investor heterogeneity. In our baseline model, the financial market consists of a risk-free bond and a stock issued by a firm. The firm is characterized by two private independent attributes that investors value: $\tilde{x}$ and $\tilde{g}$, which could represent the firm’s cash dividend and carbon footprint for example. Two types of investors trade in the market: type-\(N\) and type-\(G\). Type-\(N\) investors only care about the $\tilde{x}$ attribute, while type-\(G\) investors care about both $\tilde{x}$ and $\tilde{g}$. A monopolist information intermediary collects signals about $\tilde{g}$, designs a score, and sells it to investors, à la Admati and Pfleiderer (1986). The firm can privately and costly manipulate the signals that are used to produce the score, with the goal of maximizing its equilibrium stock price in the short run.\(^1\)

In equilibrium, the decisions of investors, firm, and information seller interact in rich ways. Investors must factor in the firm’s potential for signal manipulation as well as the information seller’s incentive for profit maximization when assessing the informativeness of the score sold by the information seller. How responsive investors (and the stock price) are to the score is a major factor in the firm’s incentive to manipulate the signal. Finally, the

\(^1\)For example, social media reactions towards new products are being used to predict company performances. A company could influence such sentiment in various ways, from paying “influencers” to market their products to using “bots” to post fake reviews.
firm’s tendency for signal manipulation and investors’ willingness to pay will influence how the information seller designs and prices the score.

We fully characterize the market equilibrium, the firm’s score manipulation strategy, and the information seller’s optimal design and pricing of the score. Several interesting results emerge. While the average degree of signal manipulability can change the level of the score (e.g., stronger manipulability could lead to higher score inflation), it does not alter the informativeness of the score or the equilibrium stock price. This is because investors already anticipate the average manipulation of the score and factor it into their decisions.

Instead, the key variable of concern about information manipulation is the uncertainty about signal manipulability across firms. Such uncertainty is a new source of noise in our model. Intuitively, higher uncertainty about manipulability should make the score and stock price less informative. Whether this leads the information seller to raise or lower the price of the score will depend on how the informativeness of the score changes relative to that of the stock price. However, this intuition could become invalid when we take into account the fact that the information seller also controls the design of the score.

As Admati and Pfleiderer (1986) show, a monopolistic information seller may want to add extra noise to a score to prevent it from becoming too revealing; otherwise the stock price will become highly informative, driving down investor demand for the score. In our model, higher uncertainty about manipulability would discourage the information seller from adding more noise to the score, which leads to two possibilities. On the one hand, in an unconstrained equilibrium, an increase in the uncertainty about manipulability is exactly offset by a reduction in noise added by the information seller, leaving the informativeness of both the score and stock price unchanged, and the price of the score remains constant as well. We refer to this as the “manipulation irrelevance” result. On the other hand, when the uncertainty about manipulability is sufficiently high, the information seller does not want to add any extra noise anymore. Instead, they would adjust the price of the score to maximize profit. In this case, the informativeness of the score and the stock price will both decline when the uncertainty about manipulability increases.

We then examine the optimal design of the score in the setting with multi-dimensional
signals. Consider first the score that is the most accurate predictor of $\tilde{g}$ (in the MSE sense). In the absence of manipulation, the most informative score is a weighted average of the individual signals that minimizes the total variance. With the possibility of manipulation, the best predictor needs to balance the noise level of individual signals (and their covariance structure) against their uncertainty about manipulability (the average level of manipulability is again irrelevant). Importantly, the relative contribution of the uncertainty about manipulability is endogenous. It depends on the firm’s incentive to manipulate individual signals, which in turn depends the sensitivity of the stock price to the score. As a result, not only do the signals’ statistical properties matter, but also the uncertainties about their manipulability and the investor composition.

Next, recall that the goal of the information seller is not to design the most informative score, but to maximize profit. This makes the optimal weights deviate from the ones that are the most informative, again to avoid making the stock price too informative. In fact, the optimal weights are generally not unique. This channel suggests that different information sellers could produce different scores even with the same set of signal inputs. It offers new perspectives on the empirically documented disagreements in credit ratings (Cantor and Packer, 1997) or ESG ratings (Chatterji et al., 2016; Berg et al., 2022) across agencies.

We also examine whether the information market helps allocate stocks with higher expected values for characteristic $\tilde{g}$ towards investors who value it more. In the baseline model where type-$G$ investors value the characteristic $\tilde{g}$, despite the presence of manipulation, we show that the stronger these investors’ preferences for $\tilde{g}$, the higher the ex-ante expected value for $\tilde{g}$ in their portfolio holdings. In other words, the market for information does help tilt type-$G$ investors’ portfolios towards stocks with higher $\tilde{g}$.

In the last part of the paper, we study a variation of the baseline model in which the type-$G$ investors are replaced by investors who care about the score for $\tilde{g}$, not $\tilde{g}$ itself. This assumption is meant to capture investors who face mandates that are tied to the scores for certain characteristics, not the actual characteristics. For this reason, we refer to this version of the model as the mandate model. For example, insurance companies face regulatory restrictions (from the NAIC) when investing in corporate bonds, whereby buying bonds with
lower credit ratings (which might be different from credit quality) would subject them to higher capital requirements. Similarly, an ESG-themed mutual fund could be required to maintain a minimum ESG score (which might be different from the actual ESG attributes) for its portfolio holding. Moreover, we assume that the mandate investors can invest in the stock only if they purchase the score from the information seller.

When the information seller’s score design is fixed, an increase in the fraction of mandate investors has two effects. First, the price becomes more informative about the $\tilde{g}$ attribute due to a bigger mass of investors trading based on information related to $\tilde{g}$. At the same time, the higher price sensitivity to the score gives the firm stronger incentive to manipulate the score, which hurts the informativeness of the score and equilibrium price. As a result, as the fraction of mandate investors increases, price informativeness changes in a non-monotonic manner; it initially rises but eventually decreases. This is sharp contrast with the baseline model, where more investors purchasing and trading based on the score unambiguously improve price informativeness. The reason is that the score serves purely an informational role in the baseline model.

Similarly, when investment mandates become stricter, i.e., when mandate investors put a bigger weight on the score in their preferences, we again find the non-monotonic effect on price informativeness and the alignment of mandate investors’ portfolio with true characteristics $\tilde{g}$: both the price informativeness and the ex-ante expected level of $\tilde{g}$ in the portfolio of mandate investors initially rises but ultimately decreases. These results suggest that pushing too hard for investment mandates on certain stock characteristics could backfire in the presence of information manipulation: the mandate investors may not end up holding more of the desired stocks.

Finally, the information seller’s optimal design of the score in the mandate model also differs significantly from the baseline model. In the baseline model, type-$G$ investors prefer more informative scores, but the information seller tends to add some noise to the score that is the most informative. The average degrees of manipulability for individual signals are irrelevant and do not affect the optimal weights. In the mandate model, however, mandate investors care about the level of the score but not how informative it is. This generates an
incentive for score inflation and makes the average degree of signal manipulability relevant for the score design. At the same time, the information seller tries to keep the noise in the score low because the mandate investors’ willingness to pay for the score is generally decreasing in its uncertainty. These two considerations lead to distinct tradeoffs for the information seller in the mandate model. For example, a signal that is both noisy and has high uncertainty of manipulability would be considered inferior in the baseline model, but it would receive higher weights if it has high degree of manipulability on average.

1.1 Literature

First, our paper is closely related to the literature on information sales. Building upon the seminal work of Admati and Pfleiderer (1986), which examines a monopolist information seller designing and selling payoff-relevant information to traders, we extend this framework by considering the firm’s information manipulation behavior. As a result, the information seller’s input in our baseline model is endogenous to its pricing and design of information, in contrast to the exogenous input in Admati and Pfleiderer (1986). Following their work, a growing body of recent literature investigates various settings involving monopolistic information intermediaries. For example, Bergemann and Bonatti (2015) explore the pricing problem of a data broker who provides data to facilitate marketing efforts. Segura-Rodriguez (2021) examines a profit-maximizing data broker who sells data to firms seeking to forecast different consumer characteristics. Meanwhile, Yang (2022) investigates a profit-maximizing data broker who sells data to firms with private production costs. The unique aspect of our model, in which the information intermediary’s actions (pricing and design) may endogenously influence the input of the information that the seller can collect, does not show up in these prior studies.

With the development of alternative data, there has been growing interest in understanding how a firm’s data can influence the learning dynamics of its quality. Begenau et al. (2018) and Farboodi and Veldkamp (2022) emphasize the role of data (information) in a firm’s production and lifecycle. They examine how information can affect a firm’s lifecycle, highlighting the
“data feedback role,” where increased data availability can encourage firm growth. Our focus on how data availability changes a firm’s incentives is different from their perspective. In our model, the equilibrium price is (partially) determined by the firm’s information available to the information seller, which incentivizes the firm to manipulate its data, consequently reducing the informativeness of the data and price.

Another related stream of literature focuses on information manipulation by agents (firms) with strategic concerns. Numerous studies investigate the manipulation of earnings announcements (eg., Dutta and Fan (2014); Crocker and Slemrod (2007); Laux and Laux (2009)) and the incentives of managers. Among these papers, our work is most closely related to Fischer and Verrecchia (2000), which is also based on a Gaussian setting. However, our paper differs from these studies because the information in our context is neither public nor free, making the role of the information seller crucial. Another difference is that we examine how the pricing and design of information by the information seller can alter the firm’s incentive to manipulate, while these earnings manipulation studies mostly focus on incentive contracts. Our framework is also related to the signaling model of Frankel and Kartik (2019), which also discusses the firm’s ability to manipulate its signals and thereby distort payoff-relevant information. However, our discussion on the role of the information seller does not have a counterpart in their paper. Both Ball (2019) and Goldman et al. (2022) consider an information intermediary’s problem where information can be endogenously manipulated. However, the information intermediary’s objectives in their papers significantly differ from ours, distinguishing our study from existing literature.

Our paper’s model predictions are also related to the growing literature on ESG rating divergence and greenwashing. Previous studies confirm substantial disagreements in ESG ratings (e.g., Berg et al. (2022); Chatterji et al. (2016)) due to differences in measurement and methodology. Besides, firms can strategically change their behavior to attract fund flows (e.g., Cooper et al. (2005)), while these behaviors have no real impact on firm performance, as evidenced by the greenwashing behavior documented in the ESG literature (eg., Kaustia and Yu (2021); Liang et al. (2022)). Our paper can rationalize the empirical observations of these studies. In particular, our model can explain why ESG rating agencies may choose
different methodologies and why they sometimes include features that are easy to manipulate. Our model highlights the importance of the interactions between ESG rating divergence and greenwashing activities. Both Pástor et al. (2021) and Goldstein et al. (2022) consider the financial market with ESG investors. Apart from the key difference of manipulable information in our baseline model, our Section 3 on mandate investment also addresses the possibility that ESG investment operates as a mandate in an investor’s problem rather than reflecting investor preferences. Our discussion on this part highlights different predictions of this angle of ESG investment on the ESG rating market.

2 Baseline Model

2.1 The Setup

The model has a single period, from time 0 to time 1. There are two assets traded in the financial market, a riskless bond with unlimited supply and a risky asset (the stocks of a firm) with random supply \( \tilde{s} \sim N(\bar{s}, \sigma_s^2) \). We assume that the average supply of the stock is positive, i.e., \( \bar{s} > 0 \). The riskless bond is the numeraire, with the risk-free rate normalized to zero.

The stock generates two types of “payoffs” \( \tilde{x} \) and \( \tilde{g} \) at time 1, where \( \tilde{x} \) and \( \tilde{g} \) are independent and normally distributed, with \( \tilde{x} \sim N(\bar{x}, \sigma_x^2) \) and \( \tilde{g} \sim N(\bar{g}, \sigma_g^2) \) respectively. We consider two different interpretations for \( \tilde{x} \) and \( \tilde{g} \). In the first case, \( \tilde{x} \) and \( \tilde{g} \) are the two parts of the cash-flow payoff at \( t = 1 \), with \( \tilde{g} \) representing the cash-flow component that is predictable by public signals (and \( \tilde{x} \) the orthogonal component). Examples of such signals include accounting and market information, as well as alternative data such as app downloads, social media sentiment, or retail traffic. Under the second interpretation, \( \tilde{x} \) represents cash-flow payoff at \( t = 1 \), while \( \tilde{g} \) represents the monetary equivalent of other firm characteristics that investors might care about, such as the firm’s credit rating or ESG attributes.\(^2\) The realizations of \( \tilde{x} \)

\(^2\)Consider for example, an investor who cares about not only the cash-flow payoff but also the firm’s carbon footprint. Then we can view \( \tilde{g} \) as the investor’s willingness to pay to reduce the firm’s carbon emission. For simplicity, we assume \( \tilde{x} \) and \( \tilde{g} \) are independent. The model can be extended to allow for correlation.
and $\tilde{g}$ are unknown to all market participants at time 0.

**Information seller.** As in Admati and Pfleiderer (1986), we assume that there is a monopolistic information seller who collects data produced by the firm that are potentially informative about $\tilde{g}$. The information seller aggregates the data to construct a score, $g_r$, and sells it to investors at price $\Phi$. The score is endogenously designed by the information seller. We assume full transparency in the seller’s scoring algorithm, which, in general, includes the signals used and weights given to individual signals (linear scores are optimal in our setting), as well as the potential of additional independent noise added.

For simplicity, we first consider the case of a one-dimensional signal. In the absence of any manipulation by the firm, the signal about $\tilde{g}$ takes the form $\tilde{g} + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$. Admati and Pfleiderer (1986) show that a profit-maximizing information seller might sometimes add additional noise (that is independent of $\epsilon$) to the score, and thus the optimal score can be viewed as $\tilde{g} + u$, where $u \sim N(0, \sigma_u^2)$ with $\sigma_u^2 \in [\sigma^2, \infty)$. The information seller will choose both the noise in the score $\sigma_u^2$ and the price of the score $\Phi$ (which is equivalent to choosing the fraction of investors who will buy the score) to maximize their profit. In the case of multiple signals, the information seller also decides on the optimal weights given to the different signals. We study that problem in Section 2.3.

**Investors.** Besides the noise traders responsible for the random supply of the stock $\tilde{s}$, there are two types of active investors in the financial market, which we label as $G$ and $N$, whose investor bases are $\Lambda$ and $1 - \Lambda$, respectively, with $\Lambda \in [0, 1]$. Both types of investors start with an initial wealth $W_0$ and have CARA utility. If a type-$i$ ($i \in \{G, N\}$) investor holds $\phi$ units of the stock and $l$ units of the bond, their expected utility at time 0 is represented by

\[
U_G = \mathbb{E} \left[ -e^{-A(\phi(\tilde{x} + \beta\tilde{g}) + l)} | \mathcal{F}_G \right],
\]

\[
U_N = \mathbb{E} \left[ -e^{-A(\phi\tilde{x} + l)} | \mathcal{F}_N \right],
\]

between the two, which could be more natural for certain firm characteristics.
where $A$ is the coefficient of absolute risk aversion, and $\mathcal{F}_i$ is type-$i$ investor’s information set at time 0. The key difference between the two types of investors is that $G$ cares about both $\tilde{x}$ and $\tilde{g}$, with the coefficient $\beta$ representing the importance of $\tilde{g}$ relative to $\tilde{x}$, whereas $N$ only care about $\tilde{x}$. Our reason to introduce type-$N$ investors is to allow for heterogeneity in investor preferences towards non-cash-flow characteristics, such as carbon footprint. In that case, type-$G$ investors correspond to “green investors.”³ Among the type-$G$ investors, a fraction $\lambda$ will endogenously decide to buy the score from the information seller, and the remainders (fraction $1 - \lambda$) will not.

**Firm.** The main difference of our baseline model from Admati and Pfleiderer (1986) is the possibility for firms to manipulate the signals. The firm would like to maximize its market value $p$ at $t = 0$. Intuitively, in the absence of any signal manipulation, when the score $g_r$ is higher, type-$G$ investors will expect a higher $\tilde{g}$ at $t = 1$, and they will bid up the stock price at $t = 0$. This effect creates an incentive for the firm to engage in signal manipulation to increase the stock price and reduce the cost of capital. For example, one could run sentiment analysis on product reviews posted on social media to predict revenue growth. This could provide companies with the incentive to fabricate positive reviews.⁴

Specifically, the firm can privately change the level of the signal by $\delta$ at the cost of $\frac{1}{2q} \delta^2$, where $q$ represents signal manipulability and is privately observed by the firm – the higher $q$ is, the easier it is for a firm to alter the signal. We assume that the public belief about $q$ follows $q \sim N (\bar{q}, \sigma^2_q)$. This assumption should be viewed as an approximation of $q = \max (0, \tilde{q})$ where $\tilde{q} \sim N (\bar{q}, \sigma^2_q)$, which ensures that $q$ stays positive.⁵ The cost of manipulation can include direct costs of manipulation activities as well as indirect costs, such as the expected costs of punishment if getting caught of cheating. After signal manipulation, the score generated by the information seller becomes

$$g_r = \tilde{g} + u + \delta. \quad (3)$$

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³Indeed, we will eliminate type-$N$ investors by setting $\Lambda = 1$ when interpreting $\tilde{g}$ as the predictable component of firm cash flow.

⁴As another example, see Elon Musk’s dispute with Twitter regarding its number of legitimate users before acquisition.

⁵In the appendix, we show that this approximation is accurate when $\bar{q} \gg \sigma_q$. 
Information seller chooses the scoring algorithm
The firm manipulates signals
Information seller generates a score based on reported signals
Investors decide whether to buy the score and submit demand for the stock; stock market clears
Asset payoff is realized

Figure 1: Timeline

While both types of active investors and the information seller are aware of the firm’s incentives to manipulate the signals, they do not observe \( q \) (they do know \( \bar{q} \) and \( \sigma_q \)) and thus cannot precisely forecast the amount of manipulation by the firm and remove its effect from the score entirely.

The firm’s objective is to maximize the stock price net of the cost of information manipulation,

\[
\max_{\delta} \mathbb{E} \left( p \right) - \frac{1}{2q} \delta^2.
\]  

(4)

Figure 1 shows the timeline of the model: At time 0, the information seller chooses both the noise \( \sigma_u^2 \) and the price of the score \( \Phi \). Then the firm decides on the level of manipulation \( \delta \), and investors decide whether to buy the score from the information seller. After that, all investors submit their demand functions and the price \( p \) is endogenously determined in equilibrium. At time 1, all random variables are realized, and all players collect their payoffs.

Next, we define the market equilibrium in the presence of information manipulation.

**Definition 1** (Noisy Rational Expectations Equilibrium). We define an equilibrium as \( \{ \Phi^*, \sigma_u^*, \lambda^*, p^*, \delta^* \} \), where \( \Phi^* \) is the price of the score chosen by the information seller, \( \sigma_u^* \) is the standard deviation of the noise in the score chosen by the information seller, \( \lambda^* \) is the fraction of type-G investors who opt to purchase the score, and \( \delta^* \) is the manipulation level that the firm selects, such that the following conditions hold:

- a fraction \( \lambda^* \) of the type-G investors purchase the score, while the remaining fraction \( (1 - \lambda^*) \) choose not to purchase the score, to maximize their utility (2):
the firm selects $\delta^*$ to maximize the objective (4);

- the information seller sets the price of the score $\Phi^*$ and the standard deviation of the noise in the score $\sigma_u^*$ to maximize its profit;

- the equilibrium price $p^*$ clears the market, i.e., the total demand for the stock by all active investors equals supply $\tilde{s}$.

### 2.2 Model Solution

We solve the market equilibrium with manipulation in two steps. First, we solve the partial equilibrium while taking the information seller’s strategy (noise level $\sigma_u$ and fraction of score buyers among type-$G$ investors $\lambda$) as given. The price of the score $\Phi$ is determined endogenously in this partial equilibrium. Second, we solve the information seller’s problem.

Given the fraction of score buyers among type-$G$ investors $\lambda$ and the noise of the score $\sigma_u$, we conjecture that the equilibrium price of the stock has a linear structure, i.e., there exist endogenous coefficients $a_0$, $a_r$, and $a_s$ such that

$$p = a_0 + a_r g_r + a_s \tilde{s}.$$ \hspace{1cm} (5)

Since only $g_r$ is observable (not $\tilde{g}$), the equilibrium price is linear in the score $g_r$, and $a_r$ represents the sensitivity of stock price to the score.

From (3) and (5), it is easy to show that the firm’s optimization problem (4) also has a linear solution:

$$\delta^* = a_r q,$$ \hspace{1cm} (6)

and thus the equilibrium score is

$$g_r = \tilde{g} + u + a_r q.$$ \hspace{1cm} (7)

Intuitively, controlling for the cost of manipulation $q$, the more sensitive the equilibrium
price is to changes in the score \( g_r \), the stronger the firm’s incentive to manipulate the score. Manipulation in turn has two effects on the score: it inflates the score by \( a_r \tilde{q} \) on average, and makes it less informative by adding additional noise with variance \( a_r^2 \sigma_q^2 \).

The score in (7) is uninformative about \( \tilde{x} \). Since type-\( N \) investors only care about \( \tilde{x} \) in their utilities, they will never buy the score. With CARA utility, their optimal demand for the stock is

\[
\phi^*_N = \frac{\mathbb{E}(\tilde{x}|\mathcal{F}_N) - p}{A\text{Var}(\tilde{x}|\mathcal{F}_N)}.
\]

Type-\( G \) investors face different inference problems about \( \tilde{g} \) depending on their decision to purchase the score. Those who buy the score observe both the equilibrium price \( p \) and the score \( g_r \), while those who do not buy the score only observe the equilibrium price. Their optimal demand for the stock is

\[
\phi^*_i = \frac{\mathbb{E}(\tilde{x} + \beta \tilde{g}|\mathcal{F}_i) - p}{A\text{Var}(\tilde{x} + \beta \tilde{g}|\mathcal{F}_i)},
\]

where \( i \in \{I, U\} \) represents type-\( G \) investors who buy and do not buy the score, respectively. Both (8) and (9) show that investors will trade more aggressively in response to news about stock payoff when the uncertainty about the payoff is lower.

The market-clearing condition is

\[
(1 - \Lambda) \phi^*_N + \Lambda [(1 - \lambda) \phi^*_U + \lambda \phi^*_I] = \bar{s}.
\]

Before presenting the solution, we define a few variables of interest, which also help simplify the exposition. First, we measure the informativeness of the score \( g_r \) about the payoff component \( \tilde{g} \) with the correlation \( \text{corr}(\tilde{g}, g_r) \), and define

\[
r_1 \equiv \text{corr}^2(\tilde{g}, g_r) = \frac{\sigma_g^2}{\sigma_g^2 + \sigma_u^2 + a_r^2 \sigma_q^2} \in (0, 1)
\]
which is also the $R^2$ of the regression that uses $g_r$ to predict $\tilde{g}$.\(^6\) Notice that the additional noise in the score comes from three sources: the intrinsic noise in the signals and potential noise added by the information seller, which together has total variance $\sigma_u^2$, and the uncertainty about signal manipulability, $a_r^2\sigma_q^2$.

We measure the informativeness of the stock price $p$ about $\tilde{g}$ with $\text{corr}(\tilde{g}, p)$, and define

$$r_2 \equiv \text{corr}^2(\tilde{g}, p) = \frac{\sigma_g^2}{\sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2 + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2} \in (0, 1) \tag{12}$$

which is also the $R^2$ of the regression that uses $p$ to predict $\tilde{g}$. Compared to the score, the price contains additional noise due to noise trader supply. Finally, we use

$$n \equiv \frac{\text{Var}(\beta\tilde{g})}{\text{Var}(\tilde{x} + \beta\tilde{g})} = \frac{\beta^2\sigma_g^2}{\sigma_x^2 + \beta^2\sigma_g^2} \tag{13}$$

to denote how much the predictable component $\tilde{g}$ contributes to the total variance of stock payoff for type-$G$ investors.

The following proposition summarizes the solution in the partial equilibrium when the information seller’s strategy is taken as given.

**Proposition 1.** Suppose that the information seller chooses noise level $\sigma_u^2$ and sets the price for the score such that a fraction $\lambda$ of type-$G$ investors choose to buy the score. Then there exists an equilibrium such that:

1. the firm’s manipulation strategy is $\delta^* = a_rq$,

2. the equilibrium score is $g_r = \tilde{g} + u + a_rq$,

3. the equilibrium price is $p = a_0 + a_r g_r + a_s\tilde{s}$,

\(^6\)Notice that the correlation $\text{corr}(\tilde{g}, g_r)$ is always non-negative in our setting. More generally, one would use $r_1$ to measure the informativeness of the score $g_r$ when $\text{corr}(\tilde{g}, g_r)$ could potentially be negative.
where \( a_0, a_r > 0 \) and \( a_s < 0 \) are uniquely solved by a system of equations:

\[
 n - n\Lambda \frac{\lambda}{\beta A\sigma_q^2} a_s = 1 + \frac{\sigma_u^2}{\sigma_q^2} + \left( \frac{1 - \frac{\lambda}{\beta A\sigma_q^2}}{\frac{\lambda}{\beta A\sigma_q^2}} \right)^2 \left( \frac{\frac{\lambda}{\beta A\sigma_q^2}}{1 - \frac{\lambda}{\beta A\sigma_q^2}} \right) \frac{\sigma_q^2}{\sigma_q^2} \tag{14a}
\]

\[
 a_r = \frac{n}{\frac{\lambda}{\beta A\sigma_q^2}} \left( 1 - \frac{\lambda}{\beta A\sigma_q^2} \right) \frac{\sigma_q^2}{\sigma_q^2} \left( \frac{\lambda}{n\Lambda} \right) \left( \frac{\frac{\lambda}{\beta A\sigma_q^2}}{1 - \frac{\lambda}{\beta A\sigma_q^2}} \right) \frac{\sigma_q^2}{\sigma_q^2}, \tag{14b}
\]

\[
 a_0 = \bar{x} + \frac{\lambda}{\sigma_q^2 + \beta^2\sigma_q^2(1-r_1)} + \left( 1 - \lambda \right) \frac{\beta \bar{q}}{\sigma_q^2 + \beta^2\sigma_q^2(1-r_2)} - A\bar{s} - (a_r (\bar{g} + a_r \bar{q}) + a_s \bar{s}) \tag{14c}
\]

Proof. See appendix. \(\Box\)

Notice that \( \bar{q} \), which represents the average manipulability of the signal, has no effect on either the equilibrium price or the informativeness of the score and the price. Higher \( \bar{q} \) does lead to more score inflation on average, but this effect on the equilibrium price is exactly offset by \( a_0 \). Intuitively, type-\( G \) investors care about \( \bar{g} \); when they update beliefs about \( \bar{g} \) through the score, they will take the perceived manipulation activities into account. Since the average degree of signal manipulation is fully anticipated, it does not change these investors’ learning or decision making. Instead, what matters for the informativeness of score and price is the uncertainty about manipulability across firms, as measured by \( \sigma_q^2 \). The following corollaries show how signal manipulability affects the informativeness of scores and prices.

**Corollary 1.** Suppose that the information seller chooses noise level \( \sigma_u^2 \) and sets the price for the score such that a fraction of \( \lambda \) type-\( G \) investors choose to buy the score. When \( \bar{q} \) increases, both \( a_r \) and \( \text{Var}(\delta^*) = a_r^2 \sigma_q^2 \) are unchanged. When \( \sigma_q^2 \) increases, \( a_r \) decreases and \( \text{Var}(\delta^*) = a_r^2 \sigma_q^2 \) increases.

Proof. See appendix. \(\Box\)

**Corollary 2.** Suppose that the information seller chooses noise level \( \sigma_u^2 \) and sets the price for the score such that a fraction of \( \lambda \) type-\( G \) investors choose to buy the score. Then
Uncertainty about manipulability: $\sigma_q$

Figure 2: **Informativeness of scores and prices.** In this figure, we fix $\lambda$ and $\sigma_u$, and examine how the change of uncertainty about manipulability $\sigma_q$ affects the informativeness of the score and the price (Panel A), as well as the correlation between the score and price (Panel B). We set $\Lambda = 0.7$, $\sigma_x = 0.2$, $\sigma_g = 0.3$, $\sigma_s = 0.4$, $\sigma_u = 0.1$, $A = 1.3$, $\lambda = 0.2$.

- $corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $corr(p, g_r)$ are independent of average manipulability $\bar{q}$;
- $corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $corr(p, g_r)$ are all decreasing in the uncertainty about manipulability $\sigma_q^2$.

**Proof.** See appendix.

Corollary 1 shows that higher uncertainty about signal manipulability $\sigma_q^2$ reduces the sensitivity of the price to the score, while Corollary 2 shows that higher $\sigma_q^2$ makes both the score $g_r$ and price $p$ less informative about $\tilde{g}$, and that the correlation between $p$ and $g_r$ also declines. These results are illustrated in Figure 2 under a simple calibration. For example, when $\sigma_q$ rises from 0.1 to 1, the correlation between $g_r$ and $\tilde{g}$ falls by about 14%, while the correlation between $p$ and $\tilde{g}$ falls even more, by about 25%.

To understand these results, let us consider how a rise in the uncertainty about signal manipulability changes the noise in the score. A direct effect of higher $\sigma_q^2$ is to make the score more noisy, due to the fact that the amount of signal manipulation in the score is
proportional to $q$ (holding $a_r$ constant; see (6) and (7)). An indirect effect of higher $\sigma_q^2$ is that it makes the price less responsive to the score ($a_r$ decreases) and lowers the correlation between the two, because investors’ beliefs about $\tilde{g}$ will become less responsive to the score as it becomes more noisy. This in turn reduces the firm’s incentive for signal manipulation, i.e., $\delta^*$ decreases when holding $q$ constant. However, since the direct effect is stronger, the overall uncertainty in the amount of manipulation, as captured by $\text{Var}(\delta^*) = a_r^2 \sigma_q^2$, still increases. 

This is why the score becomes a noisier predictor for payoff component $\tilde{g}$. Finally, since the information in price about $\tilde{g}$ ultimately stems from the score, it follows that the correlation between price and $\tilde{g}$ also decreases with $\sigma_q^2$.

Next, we turn to the information seller’s optimization problem, which involves determining the optimal price of the score $\Phi^*$ and the noise level $\sigma_u^2$.

A key difference between our setting and Admati and Pfleiderer (1986) is the possibility of signal manipulation, which influences the information seller’s incentive to sell the score. Selling the score to more investors can increase the informativeness of the price regarding payoff component $\tilde{g}$. However, the firm’s manipulation incentive may also increase, which makes the equilibrium score noisier and weaken the information spillover effect. The following lemma shows that the information spillover effect still dominates the manipulation incentive. Therefore, selling the score to more investors always leads to a more informative price regarding the true fundamental information.

**Lemma 1.** Suppose in equilibrium, a fraction $\lambda$ of type-$G$ investors choose to buy the score and the information seller chooses noise level $\sigma_u^2$. Then when $\lambda$ increases, the informativeness of the price regarding the fundamental information $\tilde{g}$, represented by $\text{corr}(p, \tilde{g})$, increases.

**Proof.** See appendix.

Those type-$G$ investors who buy the score must pay a cost $\Phi$ in exchange for more information about $\tilde{g}$. In equilibrium, a type-$G$ investor should be indifferent between purchasing the score or not, which leads to the following classic result in the literature regarding the price of the score.
Lemma 2. In equilibrium, let $\mathcal{F}_U$ and $\mathcal{F}_I$ be the information sets of type-$G$ investors who buy and do not buy the score, respectively. Then the equilibrium price of the score for any given $\lambda$ is

$$\Phi(\lambda) = \frac{1}{2A} \ln \left( \frac{\text{Var}(\hat{x} + \beta \hat{g} | \mathcal{F}_U)}{\text{Var}(\hat{x} + \beta \hat{g} | \mathcal{F}_I)} \right).$$  \hspace{1cm} (15)$$

Proof. See appendix. \qed

Notice that the dependence of the price of the score $\Phi$ on $\lambda$ in (15) is implicit. As the fraction of investors who choose to purchase the score changes, so does the informativeness of the price relative to that of the score, which determines the right-hand-side of (15).

Lemma 3. Suppose the information seller chooses a noise level $\sigma_u^2$ and price $\Phi(\lambda)$ such that a fraction $\lambda$ of type-$G$ investors buy the score. There exists a threshold $\sigma_u^2$, such that:

- when $\sigma_u^2 \leq \sigma_u^2$, $\Phi(\lambda)$ first increases and then decreases in $\sigma_q^2$;
- when $\sigma_u^2 > \sigma_u^2$, $\Phi(\lambda)$ monotonically decreases in $\sigma_q^2$.

Proof. See appendix. \qed

The total profit for the information seller is $\Lambda \lambda \Phi(\lambda)$. With the equilibrium price of the
Figure 3: **Price of the score and uncertainty about manipulability.** In this figure, we examine how the uncertainty about manipulability $\sigma_q$ changes the equilibrium price of the score while holding $\lambda$ and $\sigma_u$ fixed. The parameters are $\Lambda = 0.7$, $\sigma_x = 0.2$, $\sigma_g = 0.5$, $\sigma_s = 0.4$, $A = 1.3$, and $\lambda = 0.2$. The threshold noise level is $\sigma_u \approx 0.27$. In Panel A, $\sigma_u = 0.1$; in Panel B, $\sigma_u = 0.3$.

score in (15), we can state the information seller’s problem as follows:

$$
    \max_{\lambda \in [0.1], \sigma_u \geq \sigma_u^2} \lambda \frac{1}{2A} \ln \left( \frac{\text{Var} (\hat{x} + \beta \hat{g} | \mathcal{F}_T)}{\text{Var} (\hat{x} + \beta \hat{g} | \mathcal{F}_I)} \right).
$$

(16)

Before presenting the full solution to the equilibrium, we first discuss a quantity of interest, $-\frac{a_s}{a_r}$. Notice that the equilibrium stock price can be rewritten as

$$
    p = a_0 + a_r \left( g_r + \frac{a_s}{a_r} \hat{s} \right),
$$

where the term $-\frac{a_s}{a_r}$ measures the effect of the noisy supply $\hat{s}$ on stock price.\(^7\) A larger $-\frac{a_s}{a_r}$ makes the price a more noisy predictor of $\hat{g}$.

According to Proposition 1, when the information seller chooses $\lambda$ and $\sigma_u^2$, the equilibrium quantity $-\frac{a_s}{a_r}$ is solved by (14a). It is obvious that $-\frac{a_s}{a_r}$ is an increasing function of $\sigma_u^2$. For any given $\lambda$, there is a minimum level for $-\frac{a_s}{a_r}$, $M(\lambda)$, which is the solution to the following

\(^7\)Naturally, $a_r > 0$ and $a_s < 0$ and thus $|\frac{a_s}{a_r}| = -\frac{a_s}{a_r}$. 

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This is essentially the same equation as (14a) but with \( \sigma^2_u \) replaced by its lower bound \( \sigma^2 \). Thus, the noise in the equilibrium price contributed by the noisy supply is minimized when 
\[-\frac{a_s}{a_r} = M(\lambda),\]
and the information seller can effectively achieve any level 
\[-\frac{a_s}{a_r} \geq M(\lambda)\]
by raising \( \sigma^2_u \) from \( \sigma^2 \).

We now characterize the information seller’s optimal choice of score price \( \Phi^* \) and noise level \( \sigma^2_u \) in the following theorem.

**Theorem 1.** Define

\[
\tilde{\lambda} = \frac{\lambda_0 A \sigma_s}{\lambda} \sqrt{\frac{\sigma^2_u + \beta^2 \sigma^2_g}{\lambda}},
\]  

(18)

where \( \lambda_0 \) is a constant number solved in the appendix. Then the information seller’s optimal choice \( \Phi^* \) and \( \sigma^2_u \) are characterized as follows:

1. If \( \tilde{\lambda} \leq 1 \) and \( M(\tilde{\lambda}) \leq \frac{\sigma_s}{\sigma} \sqrt{n} \), then the information seller chooses \( \sigma^2_u \) satisfying

\[
\text{Var}(g_r) = \sigma^2_g + \sigma^2_u + a^2_r \sigma^2_q = n \left( \sigma^2_g + \frac{\Lambda \tilde{\lambda} \sigma_g}{A \beta \sigma_s \sqrt{n}} \right),
\]  

(19)

where \( a^*_r \) is the equilibrium \( a_r \) in the price equation and is defined by (63) in the appendix. The price \( \Phi^* \) is set such that a fraction \( \lambda^* = \tilde{\lambda} \) of type-\( G \) investors choose to buy the score.

2. If \( \tilde{\lambda} > 1 \) and \( M(1) \leq \frac{\sigma_s}{\sigma} \sqrt{n} \), then the information seller chooses \( \sigma^2_u \) satisfying

\[
\text{Var}(g_r) = \sigma^2_g + \sigma^2_u + a^{**}_r \sigma^2_q = n \left( \sigma^2_g + \frac{\Lambda \sigma_g}{A \beta \sigma_s \sqrt{n}} \right),
\]  

(20)

where \( a^{**}_r \) is the equilibrium \( a_r \) in the price equation, and is defined by (64) in the
appendix. The price $\Phi^*$ is the highest price under which all type-$G$ investors choose to buy the score ($\lambda^* = 1$).

3. Otherwise, in equilibrium, the information seller always chooses $\sigma_u^2 = \sigma^2$. The price $\Phi^*$ is the highest price under which a fraction $\lambda^* = \lambda_c$ of type-$G$ investors choose to buy the score, where $\lambda_c$ satisfies

$$\lambda_c = \arg \max_\lambda \lambda \ln \left(1 + \frac{A\beta^2 \sigma_s^2}{n\lambda \Lambda} \left(\frac{\beta}{M(\lambda)} + \Lambda \frac{\lambda}{A\sigma_s^2} + \frac{\beta \sigma_s^2}{n\sigma_s^2}\right)\right).$$

(21)

Proof. See appendix.

Theorem 1 shows that, depending on the model parameters, there are three possible equilibria. In the first unconstrained equilibrium, type-$G$ investors who buy and do not buy the score coexist, and the information seller chooses an interior noise level to add to the score. The other two equilibria are constrained. In the second possible equilibrium, the information seller sells the score to all type-$G$ investors, resulting in no ex-post heterogeneity among them. Nevertheless, the information seller still opts to add additional noise to the score. In the third possible equilibrium, the information seller does not add any noise to the score ($\sigma_u^2$ is constrained to $\sigma^2$), such that the total noise level is determined by the intrinsic noise in the signal, and a fraction $\lambda_c < 1$ of the investors buy the score.

An interesting property of the unconstrained equilibrium (Case 1 of Theorem 1) is that the equilibrium mass of score buyers $\Lambda \lambda^*$ and the price of the score are both independent of the fraction of type-$G$ investors $\Lambda$. The first part of this result immediately follows from (18). The score price, as (15) shows, depends on the ratio of posterior beliefs $\text{Var}(\bar{x} + \beta \tilde{g} | \mathcal{F}_U)$ and $\text{Var}(\bar{x} + \beta \tilde{g} | \mathcal{F}_L)$, both of which depend only on the mass of score buyers $\lambda \Lambda$ and the impact of noisy supply on the price $-\frac{a}{\sigma}$, which in turn depends on $\sigma_u^2$. Thus, in the unconstrained equilibrium, when $\Lambda$ changes, the information seller will adjust $\sigma_u^2$ such that the same $\Lambda \lambda^*$ and $\Phi^*$ maximize their profit, $\Lambda \lambda^* \Phi^*$.

In the following proposition, we examine how investors’ risk aversion $A$ affects the equilibrium.
A. Three possible equilibria

B. Two possible equilibria

Figure 4: Possible scenarios of Theorem 1. We consider how $(\Lambda, \sigma_q)$ affect the equilibrium structure. Parameters values are $\sigma_g = 0.3$, $\sigma_s = 1$, $\sigma_x = 0.2$, $A = 1.5$, and $\sigma = 0.1$. We set $\beta = 1.5$ for the left panel and $\beta = 6$ for the right panel. There is no unconstrained equilibrium in the right panel because $\lambda > 1$.

**Proposition 2.** In the unconstrained equilibrium characterized by Theorem 1, when the risk aversion $A$ increases locally,

- the fraction of type-$G$ investors who buy the score, $\lambda^*$, increases;
- the value of $a_r^*$, which represents the sensitivity of the equilibrium price to the score $(\partial p / \partial q_r)$, increases. Consequently, the firm’s incentive to manipulate the signal increases, as does the noise that signal manipulation adds to the score, $\text{Var}(\delta^*)$;
- the information seller’s choice $\sigma_u^2$ and the total variance of the score $\text{Var}(q_r)$ decrease, while the informativeness of the score about $\tilde{g}$, as measured by $\text{corr}(q_r, \tilde{g})$, increases;
- the variance of the equilibrium price $\text{Var}(p)$ decreases, and the informativeness of the equilibrium price about $\tilde{g}$, measured by $\text{corr}(p, \tilde{g})$, increases.

**Proof.** See appendix.

**Proposition 2** shows that, in the unconstrained case, the informativeness of the score and the price both rise when investors become more risk-averse. Intuitively, with higher risk-
aversion, investors are more willing to pay for information in order to reduce the uncertainty about the asset payoff, so it makes sense that a larger fraction of the type-$G$ investors choose to buy the score in equilibrium. At the same time, the portfolio rules (8)-(9) show that higher $A$ reduces investors’ demand for the stock, all else equal, which could make the price less responsive to changes in the score. It turns out that the first effect dominates, resulting in higher price sensitivity to the score $a_r^\ast$.

The higher price sensitivity to the score tends to strengthen the firm’s incentive to manipulate the signal, which will tend to make the score more noisy. However, the manipulation effect is not strong enough to overturn the effect of more type-$G$ investors acquiring the score. Furthermore, the information seller also responds by adding less noise to the score, which again makes both the score and price more informative.

The next proposition shows how the uncertainty about signal manipulability $\sigma_q^2$ affects the information seller’s strategy.

**Proposition 3.** Starting from the unconstrained equilibrium, as the uncertainty about signal manipulability $\sigma_q^2$ increases, the information seller first responds by decreasing the noise level $\sigma_u^2$ while keeping the score price $\Phi^\ast$ unchanged. Once the noise level reaches the lowest possible level $\sigma^2$, the information seller then responds by adjusting the score price.

**Proof.** See appendix. \hfill \Box

Unlike in Lemma 3, where we hold the noise level $\sigma_u^2$ and the fraction $\lambda$ of type-$G$ investors who buy the score fixed, in Proposition 3 we examine the relationship between score price and signal manipulability under optimal choices of $\sigma_u^2$ and $\lambda$. When the signal is sufficiently precise to begin with and the noise resulting from the firm’s signal manipulation is not too high, the information seller will want to add extra noise to the score to avoid making the score and the price too informative about $\tilde{g}$. This result is qualitatively similar to Admati and Pfleiderer (1986), and it has an important implication. It means that, as the uncertainty due to signal manipulation rises, initially it simply reduces the need for the information seller to add more noise. In fact, the information seller is indifferent about the fact that the signal is prone to manipulation under this scenario. Only when $\sigma_q^2$ becomes so high that the
information seller is no longer adding any noise to the score will they begin to care about
the noise added due to signal manipulation, which will reduce their profits. Such preferences
become important in the score design problem when there are multiple signals, which we
investigate next.

We finish this section by examining how type-G investors’ preferences for the payoff
component \(\tilde{g}\) affects their equilibrium portfolio holdings. In particular, does the market for
information help allocate stocks with higher expected value for \(\tilde{g}\) towards investors who value
such characteristics more? Proposition 4 below shows that the answer is yes, even in the
presence of signal manipulation (at least when \(\beta\) is sufficiently high).

Before stating the results, we first define an ex-ante measure for the expected value of \(\tilde{g}\)
in type-G investors’ stock holding,

\[
\mathbb{E}[\tilde{g}|G] = \frac{\mathbb{E}[(\Lambda (1 - \lambda) \phi_U^* + \lambda \phi_I^*)) \tilde{g})]}{\mathbb{E}[(1 - \lambda) \phi_U^* + \lambda \phi_I^*)]}
\]  

(22)

This is the dollar-weighted expected payoff \(\tilde{g}\) across type-G investors’ holdings, which shows
how much type-G investors’ investment tilts towards stocks with higher expected value for
\(\tilde{g}\). For reference, in the absence of the market for information about \(\tilde{g}\), the holdings of all
investors will have \(\tilde{g}\) randomly drawn, and thus \(\mathbb{E}[\tilde{g}|G] = \tilde{g}\).

With the market for information about \(\tilde{g}\), type-G investors will buy more of the stock
when its expected value for \(\tilde{g}\) is higher. This is especially true for those investor who buy the
score, which will further reduce the uncertainty about \(\tilde{g}\). As type-G investors start to care
more about \(\tilde{g}\), as captured by a higher \(\beta\), it should result in more type-G investors buying
the score, as well as stronger demand for stocks with higher expected \(\tilde{g}\). An offsetting force
in our model is that firms’ incentive to manipulate the signal also strengthens, which will
tend to make the score more noisy. However, as Proposition 4 shows, the overall effect is
that higher \(\beta\) still makes the portfolio of type-G investors tilting more towards stocks with
higher \(\tilde{g}\) in expectation.

**Proposition 4.** Suppose \(\sigma = 0\). In the case when \(\beta\) is sufficiently high, when \(\beta\) increases
locally, the expected \(\tilde{g}\) of the risky asset held by all type-G investors, represented by \(\mathbb{E}[\tilde{g}|G]\),
increases. Besides, in this case, the information seller will always choose \( \sigma_u = \sigma = 0 \) and set the price optimally such that \( \lambda = 1 \).

2.3 Score Design

In this section, we delve into the problem of endogenous score design. In practice, information sellers often design the scores they sell to the market by modifying the methodology used for score calculation. For instance, a firm may possess multiple relevant attributes, each associated with a specific intrinsic noise level and different degrees of manipulability. Certain attributes, such as accounting information, may be more difficult to manipulate (thanks to regulated reporting standards and auditing requirements) but contain a higher level of intrinsic noise (for example, accounting reports tend to be infrequent and backward-looking). In contrast, many signals constructed using alternative data have been shown to be more accurate predictors of firm performance in the absence of manipulation. The challenge is that these signals could also be more vulnerable to manipulation.

We consider an information seller designing a score by assigning weights to a variety of signals. These weights influence both the score’s overall intrinsic noise level and manipulability. The seller might also decide to add additional noise to the score, as we have explored in the baseline model. However, since adding noise to an existing signal is equivalent to shifting more weight towards more noisy signals, we will abstract away from the possibility of adding additional noise in this section, and instead focus on the trade-off between the intrinsic noise level and manipulability of different attributes.

We assume that a firm has \( N \) signals (or attributes), all of which are potentially informative about the true value of \( \tilde{g} \). In the absence of manipulation, the \( N \) attributes can be represented as \( \tilde{g} + u \), where \( i \in \{1, 2, \ldots, N\} \). We assume that \( u = (u_1, u_2, \ldots, u_N)^T \sim N(\mu, \Sigma_u) \), where \( \mu = E(u) \) and \( \Sigma_u \) is the covariance matrix. Next, for each attribute \( i \), the firm can increase its level by \( \delta_i \) at a cost of \( \frac{1}{2q_i} \delta_i^2 \), where \( q_i \sim N(\bar{q_i}, \sigma_{q_i}^2) \) represents the manipulability of attribute \( i \). We assume that the \( q_i \) across attributes are mutually independent and independent of all
other random variables in the model.\(^8\)

Given the normality structure, it is optimal for the information seller to use a linear combination of the \(N\) attributes to predict \(\tilde{g}\). They choose a vector \(w = (w_1, w_2, \ldots, w_N)^T\) such that \(w'1 = 1\) and thus the score is

\[
g_r = \sum_{i=1}^{N} w_i (\tilde{g} + u_i + \delta_i) = \tilde{g} + \sum_{i=1}^{N} w_i u_i + \sum_{i=1}^{N} w_i \delta_i.
\]

We still conjecture that there is a linear equilibrium price of the risky asset, i.e., there exists \(a_0, a_r\) and \(a_s\) such that

\[
p = a_0 + a_r g_r + a_s \tilde{s}.
\]  

(23)

The firm’s problem is

\[
\max_{\delta = (\delta_1, \ldots, \delta_N)} \mathbb{E} (p) - \sum_{i=1}^{N} \frac{1}{2q_i} \delta_i^2,
\]

(24)

and the optimal manipulation level for the \(i\)th attribute is

\[
\delta_i^* = w_i a_r q_i,
\]

(25)

and thus the equilibrium score is

\[
g_r = \tilde{g} + \sum_{i=1}^{N} w_i u_i + a_r \sum_{i=1}^{N} w_i^2 q_i.
\]  

(26)

Thus, we can decompose the noise in the score into two parts, the first due to intrinsic

\(^8\)Again, the normality assumption for \(q_i\) should be viewed as an approximation of \(q_i = \max(0, \tilde{q})\), where \(\tilde{q} \sim N(\tilde{q}, \sigma^2)\), which ensures that \(q\) stays positive. Instead of independence, we can also allow for general covariance structure between \(u\) and \(q\).
noise, the second due to uncertainty about signal manipulability.

\[
\sigma_u^2(w) = w^T \Sigma_u w, \\
\sigma_q^2(w) = \sum_{i=1}^{N} w_i^4 \sigma_{\tilde{q}_i}^2.
\]

Before presenting the equilibrium solution, let us first consider a related problem where the information seller tries to maximize the informativeness of the score conditional on a fraction \( \lambda \) of type-\( G \) investors buying the score. Let \( w_{\text{max}}(\lambda) \) be the optimal weights to this most-informative score,

\[
w_{\text{max}}(\lambda) = \arg \max_w \text{corr}(g_r, \tilde{g})
\]

\[
= \arg \min_w \sigma_g^2 + \sigma_u^2(w) + \left[ \frac{n}{\beta} + \frac{1}{A} \left( \frac{1-\Lambda}{\sigma_x^2} + \frac{\Lambda}{\sigma_x^2 + \beta^2 \sigma_y^2} \right) M_b(\lambda) \left( \frac{\lambda + \frac{\beta A \sigma_y^2}{nA} M_b(\lambda)}{1 + \frac{\beta A \sigma_y^2}{nA} M_b(\lambda)} \right) \right]^2 \sigma_q^2(w),
\]

where \( M_b(\lambda) \) is the analog of \( M(\lambda) \) from the baseline model (which is defined as the solution to (17)),

\[
n \sigma_g^2 + n \Lambda \frac{\lambda}{A \beta} M_b(\lambda) = \min_w \sigma_g^2 + \sigma_u^2(w) + \left[ \frac{n}{\beta} + \frac{1}{A} \left( \frac{1-\Lambda}{\sigma_x^2} + \frac{\Lambda}{\sigma_x^2 + \beta^2 \sigma_y^2} \right) M_b(\lambda) \left( \frac{\lambda + \frac{\beta A \sigma_y^2}{nA} M_b(\lambda)}{1 + \frac{\beta A \sigma_y^2}{nA} M_b(\lambda)} \right) \right]^2 \sigma_q^2(w).
\]

To maximize the informativeness of the score, one needs to minimize the total variance of all noises in the score. The optimal weights are not the ones that minimize the intrinsic noise \( \sigma_u^2(w) \), the standard objective of a forecasting problem in the absence of manipulation (minimizing the mean squared error). Nor do they minimize the amount of noise generated by signal manipulation. Instead, it strikes a balance between the two considerations. Thus, a signal that is highly informative in the absence of manipulation but shows significant
uncertainty in manipulability across firms may not receive a high weight in the score. Importantly, the degree to which uncertainty about manipulability contributes to the total noise, and thus its impact on $w_{\text{max}}$, is endogenous in this model. It depends on the price sensitivity to the score, $a_r$, which in turn depends on the investor composition, as well as the distribution of signal precision and manipulability. For example, the share of type-$G$ investors in the market $\Lambda$ and investors’ risk aversion coefficient $A$, which are neither related to the statistical properties of the signals nor their manipulability, will both have important effects on $w_{\text{max}}$.

The score design problem is further complicated by the fact that the information seller’s objective is not to maximize the informativeness of the score, but to maximize the profit. We characterize the equilibrium as follows.

**Theorem 2.** In the model of score design, the information seller’s optimal choice of $\Phi^*_b$ and $w^*$ are characterized as follows:

1. If $\bar{\lambda} \leq 1$ and $M_b(\bar{\lambda}) \leq \frac{\sigma_g}{\sigma_s} \sqrt{n}$, then the information seller chooses $w^*$ satisfying

   \[
   \text{Var}(g_r) = \sigma_g^2 + \sigma_u^2(w^*) + a_r^{*2} \sigma_q^2(w^*) = n\sigma_g^2 + n\Lambda \frac{\bar{\lambda}}{A\beta} \frac{\sigma_g}{\sigma_s} \sqrt{n},
   \]

   where $a_r^{*}$ is the equilibrium $a_r$ in the price equation and takes the same value as $a_r^*$ in **Theorem 1** (as defined by (63) in the appendix), and the price $\Phi^*_b$ is set such that a fraction of $\lambda^* = \bar{\lambda}$ investors choose to buy the score.

2. If $\bar{\lambda} > 1$ and $M_b(1) \leq \frac{\sigma_g}{\sigma_s} \sqrt{n}$, then the information seller chooses $w^*$ satisfying

   \[
   \text{Var}(g_r) = \sigma_g^2 + \sigma_u^2(w^*) + a_r^{**2} \sigma_q^2(w^*) = n\sigma_g^2 + n\Lambda \frac{1}{A\beta} \frac{\sigma_g}{\sigma_s} \sqrt{n},
   \]

   where $a_r^{**}$ is the equilibrium $a_r$ in the price equation, and is defined by (64) in the appendix, and the price $\Phi^*_b$ is the highest price that all investors choose to buy the score ($\lambda^* = 1$).

3. Otherwise, in equilibrium, the price $\Phi^*_b$ is the highest price that a fraction of $\lambda_b$ investors
choose to buy the signal where \( \lambda_b \) satisfies

\[
\lambda_b = \arg \max_{\lambda} \lambda \ln \left( 1 + \frac{A \beta^2 \sigma^2_s}{n \lambda \Lambda} \frac{1}{A_b(\lambda)} \right) .
\] (33)

The information seller chooses \( w^* = w_b(\lambda_b) \).

Proof. See appendix.

In the first unconstrained equilibrium in Theorem 2, the information seller attains the highest possible payoff. Notice also that in both Case 1 and 2, the optimal choice of scoring weights \( w^* \) differ from \( w_b(\lambda^*) \), the weights that maximize the informativeness of the score. Moreover, the optimal weight is not unique. This result is demonstrated in the following Corollary. Intuitively, the non-uniqueness of optimal weights is due to the fact that the information seller is indifferent about the various ways to add additional noise to the score that is the most informative about \( \tilde{g} \).

**Corollary 3.** In the unconstrained equilibrium characterized by Theorem 2, the solution \( w^* \) satisfying (31) is generically not unique.

Proof. See appendix.

The information seller only chooses the weights to maximize the informativeness of the score in the third equilibrium. This occurs when the level of intrinsic noise embedded in the signals is not too low. Again, the average degree of signal manipulability does not matter for the design of the score.

### 3 A Model of Investment Mandate

In the previous section, we study the market for manipulable information when the information is used by investors to forecast asset payoffs. The score provided by the information seller is valuable only because it helps improve investors’ forecasts. In some cases, an investor
may care about the score itself, even if the score is noisy or biased. This is often because the investor faces an investment mandate that depends on the score. For example, certain institutional investors are only allowed to invest in corporate bonds that have an investment grade rating. Another example is impact investing. An ESG fund might be required to hold a portfolio that exceeds a minimum ESG score. We refer to these investors as “mandate investors.” In the presence of mandate investors, the incentives of the investors and the information seller can become quite different. In this section, we consider a variation of the baseline model to study how the presence of mandate investors changes the market for manipulable information.

We still consider two types of investors in the financial market, a fraction \( (1 - \Lambda) \) of type-\( N \) investors who have the same preference as those in our baseline model, and a fraction \( \Lambda \) of type-\( M \) investors, which stand for “mandate investors.” To comply with the investment mandate, type-\( M \) investors must first acquire the score before constructing their portfolios.

As in the baseline model, both types of investors have initial wealth of \( W_0 \) and CARA utility. The preferences of the type-\( N \) investors are identical to those in (2); in particular, they only care about the payoff component \( \tilde{x} \). Mandate investors, on the other hand, care about both \( \tilde{x} \) and the level of the score \( \tilde{g}_r \). For example, fund managers may face pressure from their investors to incorporate ESG considerations into their investment processes. This pressure can come from individual investors, pension funds, endowments, and other institutional investors who are interested in aligning their investments with their values and ethical beliefs. To fulfill the investment mandate, we assume that if mandate investors do not buy the score from the information seller, they can only invest in the risk-free bond. If they buy the score, they can invest in both the stock and the risk-free bond, with time-0 utility

\[
U_M = \mathbb{E} \left[ -e^{-A(\phi(\tilde{x} + \beta \tilde{g}_r) + l)} | \mathcal{F}_M \right],
\]

(34)

where \( \phi \) is his holding of the stock, \( l \) is the holding of the risk-free bond and \( \mathcal{F}_M \) is the information set of mandate investors at time 0. The term \( -\beta \tilde{g}_r \) can be viewed as the shadow cost of investing in the firm with score \( \tilde{g}_r \). We assume \( \beta > 0 \) to highlight that mandate
investors prefer to hold high-rated assets. It is worth emphasizing that the key distinction between the preferences of the mandate investors in (34) and those of the type-$G$ investors in the baseline model (1) is that the former replaces the actual characteristic $\tilde{g}$ with the score $g_r$ in their utility. This distinction has significant implications for the model.

3.1 Model Solution

In this investment mandate model, we first consider the market equilibrium when $\hat{\sigma}_{u}^{2}$ (the variance of the intrinsic noise $\hat{u}$) and the price of the score $\hat{\Phi}$ are given, and then we consider the information seller’s optimal choice of $(\hat{\sigma}_{u}^{2}, \hat{\Phi})$. The equilibrium consists of investors’ optimal holding and the firm’s optimal manipulation strategy. Specifically, mandate investors decide whether to buy the score, and if they do, their optimal holding maximizes their expected utility function (34). Type-$N$ investors choose their optimal holding to maximize their utility function (2). The firm, anticipating the market equilibrium price $\hat{p}$, selects its manipulation level $\hat{\delta}$ to maximize their objective

$$\mathbb{E}(\hat{p}) - \frac{1}{2q}\hat{\delta}^2.$$  

(35)

We conjecture that there exists an equilibrium under which all mandate investors choose to buy the score, and the time-0 price of the stock is a linear combination of the score $\hat{g}_r$ and the noisy supply $\tilde{s}$. Given the equilibrium stock price, the optimal holdings of type-$N$ investors are

$$\phi^*_N = \frac{\mathbb{E}(\hat{x}|\mathcal{F}_N) - \hat{p}}{A\text{Var}(\hat{x}|\mathcal{F}_N)},$$  

(36)

and the optimal holdings of mandate investors are

$$\phi^*_M = \frac{E(\hat{x} + \beta\hat{g}_r|\mathcal{F}_M) - \hat{p}}{A\text{Var}(\hat{x}|\mathcal{F}_M)}.$$  

(37)
The market clear condition implies that
\[
(1 - \Lambda) \frac{E(\tilde{x}|\mathcal{F}_N) - \bar{\tilde{p}}}{AVar(\tilde{x}|\mathcal{F}_N)} + \Lambda \frac{E(\tilde{x} + \beta \tilde{g}_r|\mathcal{F}_M) - \bar{\tilde{p}}}{AVar(\tilde{x}|\mathcal{F}_M)} = \bar{s}.
\] (38)

Since the return \( \tilde{x} \) is independent of the true \( \tilde{g} \), the price cannot aggregate any useful information about \( \tilde{x} \). Therefore, we conjecture (and verify later) that
\[
E(\tilde{x}|\mathcal{F}_N) = E(\tilde{x}|\mathcal{F}_M) = \bar{x}
\]
and
\[
Var(\tilde{x}|\mathcal{F}_N) = Var(\tilde{x}|\mathcal{F}_M) = \sigma_x^2.
\]

The firm will choose to manipulate the score if the price is increasing in it, and the firm’s problem is
\[
\max_{\delta} E(\tilde{p}) - \frac{1}{2q} \hat{\delta}^2.
\] (39)

Similar to the baseline model, the equilibrium score \( \tilde{g}_r \) is characterized by
\[
\tilde{g}_r = \tilde{g} + \bar{\mu} + \hat{\delta},
\] (40)
then the firm’s problem becomes
\[
\max_{\delta} \bar{x} + A\sigma_x^2 \bar{s} + \Lambda \beta \left( \tilde{g} + \hat{\delta} \right) - \frac{1}{2q} \hat{\delta}^2
\] (41)
which implies that the optimal manipulation is
\[
\hat{\delta}^* = \Lambda \beta q.
\] (42)

It is clear that the firm will choose a higher manipulation level if there are more mandate investors, if the incentive for investing in high-rated assets is higher, or if the manipulability of the score is higher. Based on the above analysis, we obtain the following result.
Proposition 5. In the model of investment mandate, when the variance of intrinsic noise $\sigma_u^2$ and the price of the score $\Phi$ are given, there exists a threshold $\Phi > 0$, such that if $\Phi < \Phi$, there exists an equilibrium in which all mandate investors purchase the score. The firm’s manipulation level in this equilibrium is

$$\hat{\delta}^* = \Lambda \beta q,$$

(43)

the equilibrium price of the risky asset is

$$\hat{p} = \bar{x} + \Lambda \hat{g}_r - A\sigma_x^2 \bar{s},$$

(44)

and the equilibrium index is

$$\hat{g}_r = \bar{g} + \hat{u} + \Lambda \beta q.$$

(45)

If $\Phi > \Phi$, there exists an equilibrium in which mandate investors do not purchase the score and only invest in the risk-free asset, and only type-N investors invest in the stock. The firm will not engage in any signal manipulation activities.

Proof. See appendix.

Proposition 5 implies that the equilibrium price is

$$\hat{p} = \bar{x} + \underbrace{\Lambda \beta}_{\text{direct effect}} \left( \hat{g} + \hat{u} + \underbrace{\Lambda \beta}_{\text{indirect effect}} q \right) - A\sigma_x^2 \bar{s}.$$

The presence of mandate investors has a direct effect on asset prices: they tend to favor high-rated assets, which leads to an increase in the equilibrium stock price for firms with high score. This, in turn, results in a lower cost of capital for these firms. For mandate investors, the shadow cost of investing in assets with the score $\hat{g}_r$ is represented by $-\beta \hat{g}_r$, where the coefficient $\beta$ represents the incentive to hold high-rated assets. As $\beta$ increases, mandate investors have a stronger incentive to hold high-rated assets, driving up the price.
of those assets, and at the same time making the price more correlated with the score \( \hat{g}_r \). However, there also exists an indirect effect. The higher price for these high-rated assets can also incentivize firms to manipulate their score to boost their prices even further, similar to the 'greenwashing' practices observed in ESG ratings. Greenwashing activities undermine the overall informativeness of the ESG rating system, and the rating reflects more on the manipulability rather than the firm’s true level of greenness. Then the aggregate impact of an increase in mandate investment incentive, measured by \( \beta \), on the cost of capital is unclear.

Using a similar argument, the overall impact of an increase in the fraction of mandate investors, denoted by \( \Lambda \), on the cost of capital is also uncertain. On one hand, the direct effect of greater mandate investor participation makes asset prices and the score more correlated, but on the other hand, the indirect effect encourages firms to manipulate their score, which reduces its accuracy. As a result, the impact of \( \Lambda \) on the cost of capital is similar to that of the mandate investment incentive \( \beta \). The following proposition formalizes the above intuition.

**Proposition 6.** In the equilibrium that all mandate investors purchase the score characterized by Proposition 5, when the mandate investment incentive \( \beta \) increases, or when the fraction of mandate investors \( \Lambda \) increases,

1. the informativeness of the score \( \hat{g}_r \), measured by \( \text{corr}(\hat{g}_r, \hat{g}) \), decreases;

2. the correlation between equilibrium price and the score \( \hat{g}_r \), \( \text{corr}(\hat{p}, \hat{g}_r) \), increases;

3. the correlation between equilibrium price and true \( \tilde{g} \), \( \text{corr}(\hat{p}, \tilde{g}) \), first increases and then decreases.

**Proof.** See appendix. \( \square \)

An interesting result here is that, when a stricter market-wide investment mandate is implemented, i.e., when \( \beta \) is higher, the expected \( \tilde{g} \) of the risky asset that mandate investors hold, denoted by \( \mathbb{E}[\tilde{g}|M] = \frac{\mathbb{E}(\phi^*_M \tilde{g})}{\mathbb{E}(\phi^*_M)} \), does not necessarily increase.
Proposition 7. We have
\[
E[g|M] = \frac{E(\phi_M^* \tilde{g})}{E(\phi_M^*)} = \frac{\beta \sigma_q^2}{\Lambda(1-\Lambda)} + \beta (\tilde{g} + \Lambda \tilde{q}) + \tilde{g}.
\] (46)

Then \( E[g|M] \) increases in \( \beta \) when \( \beta < \sqrt{\frac{\Lambda \sigma_q^2}{\Lambda(1-\Lambda)}} \), and decreases in \( \beta \) when \( \beta \geq \sqrt{\frac{\Lambda \sigma_q^2}{\Lambda(1-\Lambda)}} \).

Proof. See appendix.

3.2 The information Seller’s Optimal Choice

Now we consider the information seller’s score design and pricing. Similar to the baseline model, in this section, we assume that the manipulability features \( (\tilde{q}, \sigma_q^2) \) are given, and solve for the information seller’s optimal choice of \( \tilde{\sigma}_u^2 \) and the price of the signal \( \tilde{\Phi} \). In Proposition 5, we’ve characterized the equilibrium under any \( \tilde{\sigma}_u^2 \).

We can show that the upper bound \( \tilde{\Phi} \) in Proposition 5 is also the maximum price that the information seller can charge, and is a function of \( \tilde{\sigma}_u^2 \). In equilibrium, the score does not provide any information about the monetary payoff \( \tilde{x} \), so type-\( N \) investors will never purchase the score. Since the fraction of mandate investors, \( \Lambda \), is constant, to maximize the total profit from score sales, the information seller’s problem is reduced to maximize the maximum price \( \tilde{\Phi} (\tilde{\sigma}_u^2) \), i.e.,

\[
\max_{\tilde{\sigma}_u^2} \tilde{\Phi} (\tilde{\sigma}_u^2).
\] (47)

The maximum price satisfies an indifferent condition, meaning mandate investors are indifferent between purchasing the score and not when the score price is at the maximum price \( \tilde{\Phi} \). This implies

\[
-e^{-A(W_0-\tilde{\Phi})} E \left[ e^{-A[\phi_M^* E(\tilde{\phi} + \tilde{\Phi}) + \frac{1}{2} A \phi_M^* \text{Var}(\tilde{\phi} | \tilde{\Phi})] - A(W_0-\tilde{\Phi})} \right] = -e^{-AW_0}.
\]

The following lemma provides a closed-form solution for the maximum price \( \tilde{\Phi} \).
Lemma 4. The maximum price of the score is

\[
\bar{\Phi} = \frac{1}{2A} \left[ \frac{\mu_z^2}{1 + \sigma_z^2} + \log \left( 1 + \sigma_z^2 \right) \right], \tag{48}
\]

where

\[
\mu_z = \frac{(1 - \Lambda) \beta \bar{q} + A \sigma_u^2 \bar{s} + (1 - \Lambda) \Lambda \beta^2 \bar{q}}{\sigma_x} \tag{49}
\]

and

\[
\sigma_z = \sqrt{(1 - \Lambda)^2 \beta^2 \left( \sigma_\bar{g}^2 + \hat{\sigma}_u^2 + \Lambda^2 \beta^2 \sigma_\bar{q}^2 \right) + A^2 \sigma_x^4 \sigma_u^2 \sigma_z^2}. \tag{50}
\]

Proof. See appendix. \qed

Since the information seller can choose any \( \hat{\sigma}_u^2 \in [\sigma^2, \infty) \), it’s clear that when the information seller chooses \( \hat{\sigma}_u^2 = \infty \), we have \( \sigma_z = \infty \), and thus the maximum price of the index is also \( \infty \). And this must be the information seller’s optimal choice. Then we have the following result:

Proposition 8. In the model of investment mandate, the information seller’s optimal choice of \( \hat{\sigma}_u^2 \) is \( \infty \).

One crucial reason of obtaining the aforementioned result is that the information seller holds the power to select a high \( \hat{\sigma}_u \) value, extending even up to infinity. What if the information seller can only add a limited amount of noise to the score?

Assume the information seller can only choose any \( \hat{\sigma}_u \in [\sigma, \bar{\sigma}] \), where \( \bar{\sigma} < \infty \). Let

\[
z = \frac{E (\tilde{x} + \beta \tilde{g}_r | \mathcal{F}_M) - \hat{p}}{\sigma_x} - \frac{(1 - \Lambda) \beta \tilde{g}_r + A \sigma_u^2 \bar{s}}{\sigma_x}.
\]

Here \( z \) represents the premium that an mandate investor earns by holding the risky asset. Then we can show that the \( \mu_z \) and \( \sigma_z \) in Lemma 4 satisfy

\[
\mu_z = \mathbb{E} (z)
\]
\[ \sigma_z^2 = \text{Var}(z). \]

The expected premium \( \mu_z \) depends on \( \bar{g} \). When \( \bar{g} \) is high, the expected premium \( \mu_z \) is also high because the score is only partially priced in the equilibrium price with a sensitivity \( \Lambda\beta \).

If there are fewer mandate investors, the sensitivity is lower, mandate investors can obtain high utility by holding high-rated assets but the price paid is not very responsive to the score, resulting in a higher premium for high-rated assets and a potentially low or the negative premium for low-rated assets.

In the appendix, we show that in equilibrium, a mandate investor’s expected utility after purchasing the score is

\[ U_M = -e^{-A(W_0-\bar{q})}E\left[e^{-\frac{z^2}{2\sigma_z^2}}\right]. \]

We can see that the expected utility of mandate investors only depends on the absolute value of \( z \), and its sign does not matter. This is because when the value of \( z \) is very low, mandate investors can still choose to short the asset and earn a high expected utility. The function \( -e^{-\frac{z^2}{2\sigma_z^2}} \) is flat when the absolute premium \( |z| \) is high but becomes steep when \( z \) is close to zero.

In the appendix, we also show the maximum price of the signal \( \tilde{\Phi} \) can also be rewritten as

\[ \tilde{\Phi} = \frac{-1}{A} \ln \mathbb{E}\left[e^{-\frac{z^2}{2\sigma_z^2}}\right]. \]

In the information seller’s problem (47), by changing \( \tilde{\sigma}_u^2 \), the information seller can alter the distribution of the premium \( z \). In particular, \( \tilde{\sigma}_u \) only impacts the variance of \( z \) but not the mean \( \mu_z \). If the absolute average premium \( |\mu_z| \) is high, the information seller will choose a low variance \( \sigma_z^2 \) to avoid low utility when \( z \) is close to zero. However, when \( |\mu_z| \) is low, the rating agency will choose a high variance \( \sigma_z^2 \) to ensure investors are not trapped in the low utility region where \( |z| \) is close to zero. The complete solution to the information seller’s problem (47) is characterized by the following proposition.

Proposition 9. If the information seller can choose any \( \tilde{\sigma}_u \in [\underline{\sigma}, \bar{\sigma}] \). The solution to the
information seller’s problem (47), \( \hat{\sigma}_u^* \), is the following: there exists \( \bar{q}_1 \leq \bar{q}_2 \), such that if \( \bar{q} \in (\bar{q}_1, \bar{q}_2) \), then \( \hat{\sigma}_u^* = \sqrt{\bar{q}} \); if \( \bar{q} \in (q_1, q_2) \), then \( \hat{\sigma}_u^* = \sqrt{\bar{q}} \).

**Proof.** See appendix.

The informativeness of the score is measured by

\[
\text{corr} (\hat{g}_r, \hat{g}) = \frac{\sigma_g^2}{\sigma_g^2 + \hat{\sigma}_u^2 + \Lambda^2 \beta^2 \sigma_q^2}
\]

which is a decreasing function of \( \hat{\sigma}_u^2 \). According to Proposition 9, when the expected value \( \bar{g} \) falls within an intermediate region \( (\hat{g}_1, \hat{g}_2) \), the information seller has the incentive to choose the most noisy score by selecting \( \hat{\sigma}_u^* = \bar{g} \). This is because the information seller’s objective is equivalent to maximizing the mandate investor’s utility after purchasing the score, which is the highest when the ex-post score \( \hat{g}_r \) is either very high or very low. When \( \bar{g} \) is very high, the ex-post score \( \hat{g}_r \) tends to be high, mandate investors obtain high utility by holding the risky asset, because in this case, the premium \( z \) tends to be high. When \( \bar{g} \) is very low, the ex-post score \( \hat{g}_r \) tends to be low and the information seller chooses to short the asset for high utility. Based on this observation, the information seller’s objective is to generate the extreme value of the score. So when the expected value \( \bar{g} \) is intermediate, the information seller can make the distribution of \( \hat{g}_r \) more dispersed by choosing the maximum \( \hat{\sigma}_u^* = \sqrt{\bar{q}} \). On the other hand, if the expected value \( \bar{g} \) is already an extreme value (either very high or very low), the information seller will lower the noise in \( \hat{g}_r \) by selecting \( \hat{\sigma}_u^* = \sqrt{\bar{q}} \).

The above discussion implies that the widely-used investment mandate based on scores is potentially problematic not only because firms have the incentive to engage in manipulation, but also because information sellers may have an incentive to intentionally choose noisy measures.

Proposition 9 highlights the possibility that the information seller may deliberately choose a noisy score. We are also interested in how investment mandate \( \beta \) changes the information seller’s incentive of designing noisy scores. The following Proposition shows that under certain conditions, increasing investment mandate \( \beta \) can incentivize the information seller to reduce
Proposition 10. Suppose \( \bar{g} > 0 \) and the information seller can choose any \( \hat{\sigma}_u^2 \in [\bar{\sigma}, \bar{\bar{\sigma}}] \). When the investment mandate is \( \beta \), let \( \hat{\sigma}_u^*(\beta) \) be the solution to the information seller’s problem (47). Then for any \( 0 < \beta_1 < \beta_2 \), we have there exists \( \Lambda \in (0,1) \), such that if \( \Lambda > \Lambda \), we must have

\[
\hat{\sigma}_u^*(\beta_2) \leq \hat{\sigma}_u^*(\beta_1).
\]

Proof. See the Appendix.

3.3 Score Design

In this part, we consider the information seller’s score design problem in the investment mandate model. Similarly to our formulation in Section 2.3, we consider an information seller who designs a score by adjusting the weights to a variety of signals. For the same reason discussed in Section 2.3, we abstract away from the possibility of adding additional noise in this section.

We assume that the firm has \( N \) signals (or attributes), all of which are informative about the true value of \( \bar{g} \). We borrow the notations used in Section 2.3, specifically, we assume that the vector of intrinsic noises is \( \hat{u} \equiv (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N)^T \sim N(\tilde{\mathbf{u}}, \Sigma_{\tilde{\mathbf{u}}}) \), and the information seller’s score is a linear combination of the \( N \) attributes, which is represented by the weight vector \( \mathbf{w} = (w_1, w_2, \ldots, w_N)^T \) that satisfies \( \mathbf{w}' \mathbf{1} = 1 \). Motivated by the scoring rule used in practice, we assume that the weights also need to satisfy \( w_i \geq 0 \) for all \( i \). For each attribute \( i \), the firm can increase its level by \( \hat{\delta}_i \) at a cost of \( \frac{1}{2\hat{q}_i} \hat{\delta}^2 \), where \( q_i \sim N(\hat{q}_i, \sigma_{q_i}^2) \) represents the manipulability of attribute \( i \). We assume that the \( q_i \) values are mutually independent and independent of all other random variables in the model.\(^9\) Based on these assumptions, the

\(^9\)Again, the normality assumption for \( q_i \) should be viewed as an approximation of \( q_i = \max(0, \hat{q}_i) \), where \( \hat{q}_i \sim N(\hat{q}_i, \sigma_{\hat{q}_i}^2) \), which ensures that \( q \) stays positive. Instead of independence, we can also allow for a general covariance structure between \( \hat{u} \) and \( q \).
score is
\[ \hat{g}_r = \sum_{i=1}^{N} w_i \left( \tilde{g} + \hat{u}_i + \hat{\delta}_i \right) = \tilde{g} + \sum_{i=1}^{N} w_i \hat{u}_i + \sum_{i=1}^{N} w_i \hat{\delta}_i. \]

We still conjecture and verify that there is a linear equilibrium price of the risky asset. Since the intermediate steps resemble our proofs in Section 2.3, we skip the intermediate steps and just present the final results. In equilibrium, we show that the firm’s optimal manipulation level for the \( i \)th attribute is
\[ \delta^*_i = w_i \Lambda \beta q_i, \] (52)
and the equilibrium score is
\[ \hat{g}_r = \tilde{g} + \sum_{i=1}^{N} w_i \hat{u}_i + \Lambda \beta \sum_{i=1}^{N} w_i^2 q_i. \] (53)

Then we can decompose the noise in the score into two parts, the first due to intrinsic noise, the second due to signal manipulation. Their variances are respectively
\[ \sigma^2_u(w) = w^T \Sigma_{\hat{u}} w, \] (54)
\[ \sigma^2_q(w) = \sum_{i=1}^{N} w_i^4 \sigma^2_{q_i}. \] (55)

Then the equilibrium in this part is similar to the model with one signal in Section 3.1, where \( \tilde{q} = \sum_{i=1}^{N} w_i q_i \) and \( \sigma^2_q = \sigma^2_q(w) \), and the variance of the intrinsic noise is \( \sigma^2_u = \sigma^2_u(w) \). Based on this observation, the information seller’s problem is
\[ \max_{w'1=1, w_i \geq 0} \bar{\Phi} = \frac{1}{2 \Lambda} \left[ \frac{\mu_z^2}{1 + \sigma^2_z} + \log \left( 1 + \sigma^2_z \right) \right], \] (56)
such that
\[ \mu_z = \frac{(1 - \Lambda) \beta \tilde{g} + A \sigma^2_{z\tilde{s}} + (1 - \Lambda) \Lambda \beta^2 \sum_{i=1}^{N} w_i \tilde{q}_i}{\sigma_x} \]
Figure 5: Illustration of Proposition 11. We consider a two-signal case with weights \((w, 1 - w)\). The first signal has \(\bar{q}_1 = 7\) and \(\sigma_{q_1} = 3\), and the second signal has \(\bar{q}_2 = 3\) and \(\sigma_{q_2} = 1\). The blue line plots the info seller’s profit when choosing difference weight \(w\), and the red line is the weight maximizing the informativeness of the score. Other parameters used are: \(\Lambda = 0.5\), \(A = 1.5\), \(\beta = 1.2\), \(\sigma_x = 0.3\), \(\bar{s} = 1\), \(\sigma_g = 0.3\), \(\sigma_{\hat{u}_1} = \sigma_{\hat{u}_2} = 0\).

and

\[
\sigma_z = \sqrt{(1 - \Lambda)^2 \beta^2 \left( \sigma_g^2 + \sigma_{\hat{u}}^2 (w) + \Lambda^2 \beta^2 \sigma_q^2 (w) \right) + A^2 \sigma_x^4 \sigma_s^2}.
\]

Let \(\hat{w}^*\) be the solution to the above problem. To establish a benchmark, let’s introduce the weight \(\hat{w}_{\max}\) that maximizes the informativeness of the score, i.e.,

\[
\hat{w}_{\max} = \arg \max_w \text{corr} \left( \hat{g}_r, \hat{g} \right),
\]

which is equivalent to minimizing the \(\sigma_z\) in the above problem:

\[
\hat{w}_{\max} = \arg \min_w \sigma_z = \arg \min_w \sigma_g^2 + \sigma_{\hat{u}}^2 (w) + \Lambda^2 \beta^2 \sigma_q^2 (w).
\]

Proposition 11. Suppose that all weights are strictly positive in \(\hat{w}_{\max}\), then

\[
\hat{w}^* \neq \hat{w}_{\max}
\]

if not all \(\bar{q}_i\) have the same value, i.e., there exist \(i \neq j\), such that \(\bar{q}_i \neq \bar{q}_j\).
Figure 5 is an example of Proposition 11 with only two signals. So, the weight $w$ is reduced to a scaler $\bar{w}$. In this example, it’s clear that the information seller tends to overweight the signal with a higher average manipulability $\bar{q}$.

**Lemma 5.** For any $\epsilon > 0$, there exists a constant $C$, such that when $\min_i \{\bar{q}_i\} \geq C \max_i \{\sigma_{q_i}\}$, in equilibrium, the probability that mandate investors short the risky asset is below $\epsilon$.

## 4 Conclusion

We study how investors, firms, and information sellers interact in a market with manipulable information. Our model builds on the framework of Admati and Pfleiderer (1986) and introduces two new features: information manipulability and investor heterogeneity. In the baseline model where investors care about actual characteristics, the average degree of signal manipulability has no effect on the equilibrium, whereas the uncertainty about signal manipulability plays a key role. Its contribution depends on firms’ incentive to manipulate the signals that are used to generate the score, which in turn depends on the equilibrium price sensitivity to the score. The optimal design of the score in this setting weights the precision of different signals against the endogenous uncertainty from manipulation. The introduction of mandate investors, who care about the scores on the characteristics and not the characteristics themselves, generates a new incentive for information sellers to inflate the scores. Pushing too strongly on the mandate could lead to reduction in the informativeness of the score and the equilibrium price, and could even result in mandate investors holding less of the desired stocks.
References


