Dynamic Information Acquisition and Asset Prices *

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Abstract

This paper studies the dynamics of information acquisition and uncertainty in a noisy rational expectations model. Investors choose to acquire most information at times when uncertainty and risk premia are high; this choice feeds back and endogenously reduces subsequent uncertainty. Within the model, uncertainty can be measured directly from risk-neutral variance—analogous to the VIX index—so this translates into the concrete prediction that risk-neutral variance mean-reverts rapidly following spikes in volatility, as is observed empirically. The cyclicality of information acquisition depends on the skewness of the underlying asset: if the market is negatively skewed, market-level information acquisition is countercyclical. Conversely, information acquisition and risk premia are high following good news for positively skewed assets such as individual stocks, which gives rise to momentum in the stock market.

Keywords: Dynamic information acquisition, Uncertainty, Investor attention, Risk-neutral variance

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1. Introduction

Investors acquire information on asset fundamentals when they trade in financial markets. Information acquisition determines portfolio choices and thus affects asset prices and the uncertainty that investors face. Conversely, uncertainty about asset payoffs changes investors’ incentives to acquire information. This interplay between information acquisition and uncertainty is central to understanding how asset prices dynamically evolve. The literature on information acquisition and asset prices typically treats information acquisition as a one-off decision. This paper investigates the endogenous dynamics of information acquisition and uncertainty in a noisy rational expectations model.

Within this model, uncertainty can be measured directly from the risk-neutral variance of the asset payoff, analogous to the volatility index VIX. In periods of high uncertainty, investors acquire more information. This choice feeds back and endogenously reduces subsequent uncertainty. As a result, the risk-neutral variance mean reverts rapidly following spikes in volatility, in line with empirical evidence. The cyclicality of information acquisition depends on the skewness of the asset payoff: market-level information acquisition is countercyclical because aggregate stock market displays negative skewness; in contrast, firm-level information acquisition is procyclical because individual stocks are positively skewed. Following good news for individual stocks, information acquisition and risk premia are high, which gives rise to momentum in the stock market.

I start by analyzing an economy with a single risky asset. Most findings and predictions extend to a multiple-asset setup that I later consider. The economy is populated by a continuum of ex-ante identical investors. Exogenous noisy supply of the asset prevents the price from fully revealing the asset’s final payoff. Knowledge about the payoff is gradually acquired over multiple time periods and comes to investors in the form of a stream of private signals. The sources of private information are different across investors, as in Hellwig (1980). I use investor attention to represent the precision of the private signal. Information is costly to investors and the cost is increasing and convex in attention.

In this paper, attention represents both the effort in gathering information and the amount of information acquired by investors in a given period.\footnote{Attention in this paper represents information acquisition. I use attention and information acquisition interchangeably throughout the paper.} This study focuses on the variation in investors’ attention levels over time rather than the static allocation of attention across assets. Investors acquire less information in aggregate when the market is devoid of profitable investment opportunities.
In this economy, investor attention is endogenously determined by the level of uncertainty. I show that uncertainty is represented by the risk-neutral variance of the asset’s terminal payoff, which quantifies the value (in utility terms) of a marginal piece of information. It can therefore be directly measured, in principle, from option prices, on similar lines to the construction of the volatility index, VIX.

Investors also learn from past prices. This generates rich dynamics as the public information set, which consists of the entire history of asset prices, grows over time. I show that four state variables summarize the public information and that these state variables have a natural economic interpretation (discussed further in Section 3.2). The four state variables jointly define a system of PDEs, which characterize equilibrium and can be solved numerically.

I allow the payoff of the risky asset to have an arbitrary distribution. Non-normality is more plausible empirically. It gives rise to endogenously fluctuating uncertainty and to comovement between asset prices and information acquisition. If the payoff is normally distributed, uncertainty no longer varies across states because the risk-neutral variance becomes a deterministic function of time.

A simple illustration of the dynamic interactions between information acquisition and asset prices is as follows. Suppose that the economy enters a period of high uncertainty about asset payoffs. If the level of information acquisition does not change, then the volatility of asset returns increases, and remains uniformly high during the high-uncertainty period. Agents, however, respond to the high uncertainty by acquiring more information. This causes asset returns to be even more volatile in the early stages of the high-uncertainty period, as agents learn more information. Returns become less volatile in the later stages, as learning gradually reduces payoff uncertainty. Thus, dynamic information acquisition causes return volatility to vary more over time and to be less persistent.

The model generates a rich set of implications supported by empirical observations. First, uncertainty is mean-reverting. High uncertainty induces investors to devote more attention to the asset, in turn creating downward pressure on uncertainty itself. As a result, the model explains why peaks in VIX are usually followed by a rapid decline.²

Second, the expected return of an asset is increasing in investor attention. When attention is high today, investors gather information at a rapid pace, leading to a rapid

²Martin (2017) find that the leading models including Campbell and Cochrane (1999), Bansal and Yaron (2004), Bollerslev, Tauchen, and Zhou (2009), and Wachter (2013) cannot explain the relatively low autocorrelation of VIX.
reduction in uncertainty and the amount of risk that this asset entails. Then, the asset becomes less risky tomorrow and enjoys a lower risk premium tomorrow and beyond. This quick reduction in tomorrow’s risk premium compared to today’s implies a high expected return for the asset.

A growing list of empirical evidence supports this prediction. Da, Engelberg and Gao (2011) find that an increase in Google Search Volume Index predicts higher stock prices in the next two weeks. Lou (2014) documents that increased advertising spending is associated with more attention and a rise in abnormal stock returns. Lee and So (2017) show that abnormal analyst coverage predicts improvements in firms’ fundamental performance. Because attention is determined by uncertainty, the expected return of the asset is also positively associated with uncertainty. Martin (2017) find that the risk-neutral variance predicts the return of the market at horizons from one month to one year.

Third, the cyclicality of information acquisition depends on the skewness of the underlying asset. When the distribution of the payoff is right-skewed, investors are more excited about potential upside gains. Therefore they devote more attention to the asset when the price is high. Conversely, when the distribution is left-skewed, investors are more worried about potential losses in the downside and acquire more information when the price is low. Aggregate stock market displays negative skewness and individual stocks display positive skewness (e.g., Bakshi, Kapadia and Madan (2003); Albuquerque(2012)). Therefore, attention is procyclical for firm-specific information and countercyclical for market-wide information. Hong, Lim and Stein (2000) find that negative firm-specific news travels more slowly compared to positive news. In contrast, Garcia (2013) documents that investors react more strongly to business cycle news at times of recession.

I extend the model to multiple assets to explain momentum in the stock market. Investors trade several individual stocks. Each stock’s payoff consists of a common market component and a stock-specific (idiosyncratic) component. Information acquisition on this idiosyncratic component determines the dynamics of the stock’s excess return. Stocks that performed well relative to the market in the past are likely to attract more attention and thus continue to generate higher expected excess returns in the future.

**Related Literature**

This paper relates to extensive literature on information acquisition in financial markets, initiated by Grossman and Stiglitz (1980) and Verrecchia (1982) and developed by Holden and Subrahmanyam (2002), Mendelson and Tunca (2004), Veldkamp(2006), Huang and Liu (2007), and Andrei and Hasler (2014), among others. The model is in the spirit
of Verrecchia (1982). It allows for a group of ex-ante identical investors to learn from both prices and diverse private information.

Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) study how mutual fund managers allocate attention across different assets and used the state of the business cycle to predict information choices. Banerjee and Breon-Drish(2017) consider a strategic trader who optimizes the time to acquire costly information about an asset's payoff in a Kyle(1985) framework. They show that equilibrium with smooth trading and a pure acquisition strategy cannot exist when the market maker cannot observe acquisition. In contrast to the above papers, where the incentive to acquire information is affected by exogenous business cycle variations or public news, investor attention in this model is determined by the endogenously generated risk-neutral variances.

The explanation of momentum also differs from existing behavioral and rational models. Hong and Stein (1999) explain underreaction and overreaction in asset markets through two groups of bounded rational investors and the assumption that information diffuses gradually across the population. Andrei and Cujean (2017) build a rational-expectations model where investors use word-of-mouth communication to acquire information and show that price exhibits momentum when information flows at an increasing rate. In both cases, momentum arises because signals first observed by a small group of investors are subsequently released to a larger group. This paper's explanation relies on no such channel and is based on the interaction between attention and risk premium.

This study also contributes to the analysis of rational expectations equilibrium with general payoff distributions. Breon-Drish (2015) relax the normality assumption and proved the existence of and characterized the equilibrium for a class of models that nests the standard Grossman and Stiglitz (1980) and Hellwig (1980) setups. Chabakauri, Yuan and Zachariadis (2017) analyze asset prices in both complete and incomplete markets for realistic multi-asset economies with non-normal payoff distributions. Previous studies employed static setups but this paper works with a dynamic one.

The paper is organized as follows. Section 2 introduces the model setup. Section 3 solves investors' portfolio and attention choice problems and characterizes the equilibrium. Section 4 illustrates model predictions and empirical implications. Section 5 extends the model to a multi-asset setup. Section 6 concludes.
2. Model

The economy features a single risky asset and a risk-free asset. It is populated by a continuum of ex-ante identical investors who actively acquire private information on the asset’s final payoff. To facilitate the exposition, I start with a discrete-time economy with dates \( t = 0, t, 2t, \ldots, T \), and later take a continuous-time limit when I characterize the equilibrium in Section 3.3.

The main differences with a standard rational expectations model (e.g. Hellwig(1980)) are that information acquisition is endogenous and that the payoff of the risky asset is not necessarily normally distributed.

**Assets**

The risk-free asset is the numéraire in this economy, and its price is normalized to 1 for all dates. The interest rate is equal to 0 for all periods. The risky asset realizes a liquidating payoff \( y \) at the final date \( T \) and pays no dividend between 0 and \( T - \Delta t \). The distribution of the terminal payoff \( G(y) \) is not restricted to normal distribution. I assume that the moment generating function of this distribution \( M_y(\theta) \) exists for any \( \theta \). This technical assumption guarantees the existence of investors’ asset demands in equilibrium.

\[
M_y(\theta) = \mathbb{E}[e^{\theta y}] = \int_{-\infty}^{\infty} e^{\theta y} dG(y) < \infty, \quad \theta \in \mathbb{R}. \tag{1}
\]

Assets are traded at dates \( t = 0, \Delta t, 2\Delta t, \ldots, T - \Delta t \). Let \( p_t \) represent the price of the risky asset at date \( t \). For the period between date \( t \) and \( t + \Delta t \), the return on the risky asset is \( p_{t+\Delta t} - p_t \). Noisy supply of the risky asset prevents the price from fully revealing the final payoff. I assume that this supply \( z_t \) follows a random walk and its increment \( z_{t+\Delta t} - z_t \) is normally distributed with a mean of zero and variance of \( \sigma_z^2 \Delta t \):

\[
z_{t+\Delta t} - z_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_z^2 \Delta t). \tag{2}
\]

**Information Acquisition**

Investors start with no information about the payoff of the risky asset and gradually acquire a stream of private signals about this payoff. Investor attention determines the precision of these signals.

Suppose investor \( i \in [0,1] \) devotes attention \( a_{it} \) to payoff \( y \) for the period between date \( t \) and date \( t + \Delta t \). New private information for this investor in this period is represented by a signal \( s_{t+\Delta t} \), which communicates \( y \) perturbed by a normal noise with precision \( a_{it} \Delta t \).
Investors acquire different pieces of information and their sources of private information are independent, as in Hellwig (1980) and Verrecchia (1982). In this setup, the noises in signals are independent both across time and across investors:

\[ s_{t+\Delta t} = y + \epsilon_{t+\Delta t}, \quad \epsilon_{t+\Delta t} \sim \mathcal{N}(0, (a_{it}\Delta t)^{-1}). \]  

(3)

New information comes at a cost that is increasing and convex in attention. This cost takes the form of \( C(a_{it})\Delta t \), where \( C(a_{it}) \) is a function with continuous first and second order derivatives. Marginal cost of attention \( C'(a_{it}) \) is increasing in \( a_{it} \).

Investors also learn from past prices. The price history up to the current date \( p_0, p_{\Delta t}, \ldots, p_t \) represents all available public information. Investor \( i \)’s information set at date \( t \) consists of her private signals \( s_{\Delta t}, s_{2\Delta t}, \ldots, s_{it} \) and the public information set. Investors are rational and use all available private and public information to update their beliefs about the asset payoff and future investment opportunities.

**Preference and Investor Optimization**

Investor \( i \) is endowed with initial wealth \( W_{i0} \) at date 0. At date \( t \), she allocates her wealth \( W_{it} \) to \( \theta_{it} \) units of risky asset and \( W_{it} - \theta_{it}p_t \) units of the risk-free asset. She also decides at this date how much attention \( a_{it} \) to devote to payoff \( y \) for the period between \( t \) and \( t + \Delta t \). It is worth noting that the corresponding private signal \( s_{i,t+\Delta t} \) arrives at date \( t + \Delta t \) and could only be incorporated in investor \( i \)'s portfolio choice from that date onwards.

All investors have constant absolute risk aversion (CARA) preference with the risk aversion parameter \( A \). They make portfolio and attention choices \( (\theta_{it}, a_{it}) \) to maximize expected utility over the terminal wealth:

\[ \max_{\theta_{it},a_{it}} \mathbb{E}_t[U(W_{iT})], \quad U(W_{iT}) = -e^{-AW_{iT}}, \]  

(4)

subject to self-financing budget constraints, given by:

\[ W_{t+\Delta t} = W_{it} + \theta_{it}(p_{t+\Delta t} - p_t) - C(a_{it})\Delta t. \]  

(5)

**Price and Market Clearing**

Equilibrium price \( p_t \) is determined by market clearing:

\[ \int_{i=0}^{1} \theta_{it}di = z_t. \]  

(6)

where the left-hand side represents the aggregate demand of the risky asset and the right-hand side, its supply.
Definition of Equilibrium

The definition of equilibrium is standard. Investors make the optimal portfolio and attention choices and the market clears.

**Definition 1.** The equilibrium is a set of risky asset prices $p_t$ and portfolio and attention policies $(\theta_{it}, a_{it})$ that solves the optimization problem (4) for each investor and satisfies market clearing condition (6).

3. Equilibrium

In this model, the asset payoff $y$ and the noisy supply of the risky asset $z_0, z_{\Delta t}, \ldots, z_{T-\Delta t}$ are exogenously given. Price $p_t$, attention $a_{it}$ and uncertainty (measured by the risk-neutral variance of the asset payoff) are endogenously determined.

I characterize the equilibrium in a three-step process. First, I solve investors’ portfolio and attention choices. Second, I use the market clearing condition to express the asset demand in terms of the exogenous variables noisy supply and asset payoff. The time series of asset demands is informationally equivalent to the time series of past asset prices. I further define state variables that summarize the information content of these time series. Third, I derive a system of equations for the price and the risk-neutral variance.

3.1 Portfolio and Attention Choice

Let us first consider a suggestive three-period example. Suppose that the payoff is realized at the final date $T = 2$ and assets are traded at dates $t = 0, 1$ with a time interval of $\Delta t = 1$. I solve investors’ optimization problems through backward induction, starting from date 1 and then moving on to date 0.

Date 1 is the last trading opportunity before the realization of payoff. Any information arriving after date 1 is worthless to investors because they could no longer change their portfolios. As a result, investors will not devote any attention to the payoff for the period between date 1 and date 2.

At date 1, from investor $i$’s perspective, $y$ follows distribution $G(y|p_0, p_1, s_{i1})$. She chooses $\theta_{i1}$ to maximize expected utility:

$$\max_{\theta_{i1}} \int -\exp \left( -A\theta_{i1}(y - p_1) \right) \, dG(y|p_0, p_1, s_{i1}).$$

Since investors are infinitely small and their sources of private information are inde-
dependent, the asset price does not depend on one particular investor’s signal. This implies that conditional on payoff $y$, past prices $p_0, p_1$ are independent from signal $s_{i1}$. To put it another way, the private signal is a source of information independent of the price history. Applying Bayes theorem and noting that $s_{i1}$ is normally distributed with mean $y$ and precision $a_{i0}$, the optimization problem (7) is equivalent to:

$$\max_{\theta_{i1}} \int -\exp\left(-A\theta_{i1}(y - p_1)\right) \cdot \frac{1}{\sqrt{2\pi(a_{i0})^{-1}}} \exp\left(-\frac{a_{i0}}{2}(s_{i1} - y)^2\right) \ dG(y|p_0, p_1).$$  \hspace{1cm} (8)

The coefficient of $y$ in the above expression is $-A\theta_{i1} + a_{i0}s_{i1}$, a linear combination of asset demand and private signal. Investors’ utility maximization problems are similar despite the differences in the private signals. Taking the first-order condition, I find that $-A\theta_{i1} + a_{i0}s_{i1}$ is identical across investors. As a result, the asset demand $\theta_{i1}$ is additively separable in the signal, as in Breon-Drish (2015). It also contains a common component $\theta_1$ that only depends on the price history. Lemma 1 reports this asset demand. Proofs of all lemmas and propositions are given in the appendix.

**Lemma 1.** Asset demand $\theta_{i1}$ is the sum of the attention weighted private signal and a component $\theta_1$ that is common across investors and only depends on the prices $p_0$ and $p_1$:

$$\theta_{i1} = \theta_1(p_0, p_1) + A^{-1}a_{i0}s_{i1}. \hspace{1cm} (9)$$

At date 0, investors do not possess any private signals and have identical asset demands $\theta_{i0} = \theta_0(p_0)$ that only depend on price $p_0$. They also decide on this date how much information to acquire for each asset for the period between date 0 and date 1. A higher level of attention improves portfolio choices at date 1 and increases investors’ expected utility. Substituting in $\theta_{i0}$ and $\theta_{i1}$ and integrating over signal $s_{i1}$, I find that:

$$E_0^i[U(W_{t2})] = E_0^i\left[-\exp\left(-AW_{i0} - A\theta_{i0}(p_1 - p_0) - A\theta_{i1}(y - p_1) + A \cdot C(a_{i0})\right)\right]
= E_0^i\left[ \int \exp\left(-AW_{i0} - A\theta_{i0}(p_1 - p_0) - A\left(\theta_{i1} - A^{-1}a_{i0}s_{i1}\right)(y - p_1)
- \frac{a_{i0}}{2}(y^2 - p_1^2) + A \cdot C(a_{i0})\right) \cdot \frac{1}{\sqrt{2\pi(a_{i0})^{-1}}} \exp\left(-\frac{a_{i0}}{2}(s_{i1} - p_1)^2\right) ds_{i1} \right]
= E_0^i\left[ -\exp\left(-AW_{i0} - A\theta_{i0}(p_1 - p_0) - A\theta_1(y - p_1)
- \frac{a_{i0}}{2}(y^2 - p_1^2) + \frac{A \cdot C(a_{i0})}{\text{Value of Information}}\right) \right]. \hspace{1cm} (10)$$
In the above expression, \( A \cdot C(a_{i0}) \) represents the cost of information and \( a_{i0}/2 \cdot (y^2 - p_1^2) \) measures the expected gain in utility from a precision \( a_{i0} \) signal. Let \( E^*(X) \) represent the risk-neutral expectation of a random variable \( X \), defined by \( E[U'(W_{i2})/E[U'(W_{i2})] \cdot X] \), where \( U'(W_{i2}) \) is investor \( i \)'s marginal utility.

Differentiate date 0 expected utility \( E_0^i[U(W_{i2})] \) with respect to attention level \( a_{i0} \):

\[
C'(a_{i0}) = \frac{1}{2} A^{-1} E_0^i \left[ U'(W_{i2})(y^2 - p_1^2) \right] / E_0^i \left[ U'(W_{i2}) \right] \\
= \frac{1}{2} A^{-1} E_0^* \left[ y^2 - p_1^2 \right] \\
= \frac{1}{2} A^{-1} E_0^* \left[ \text{Var}_1^*(y) \right].
\]

Marginal cost \( C'(a_{i0}) \) is an increasing function of attention \( a_{i0} \). It is proportional to the expectation of \( y^2 - p_1^2 \) under the risk-neutral measure. Since the interest rate is zero, \( p_1 \) is equal to \( E_1^1[y] \), and thus \( E_1^1[y^2 - p_1^2] \) is equal to the risk-neutral variance of the payoff \( \text{Var}_1^*(y) \). This risk-neutral variance measures the marginal value of information for signal \( s_{i1} \) and represents the uncertainty in payoff \( y \) from the investors’ perspective. The precision of this signal needs to be decided one period ahead, at date 0. As a result, attention \( a_{i0} \) is determined by the risk-neutral expectation of next-period risk-neutral variance \( E_0^* \left[ \text{Var}_1^*(y) \right] \).

Investors are ex-ante identical and have the same prior belief and cost of information. In equilibrium, they choose the same level of attention \( a_{i0} \). The risk-neutral expectations and variances of the payoff computed from different investors’ marginal utilities are identical.

The risk-neutral variance \( \text{Var}_1^*(y) \) can be directly measured from option prices on similar lines to the construction of the volatility index \( \text{VIX}^3 \), thereby relating the unobservable information acquisition to an empirically observable measure. Allowing options to be traded will not change the equilibrium allocation. If there are no exogenous supply, investors’ demand for options will also be identical at a level of zero.

Equations (9) and (11) report investors’ portfolio and attention choices for this particular example where \( T = 2 \) and \( \Delta t = 1 \). Proposition 1 generalizes these findings to arbitrary \( T \) and \( \Delta t \).

**Proposition 1.** Let \( v_t \) represent the risk-neutral variance of the payoff at date \( t \):

\[ v_t \equiv \text{Var}_t^*[y] \quad (12) \]

\(^3\)The risk-neutral variance in this paper more closely resembles volatility index \( \text{SVIX} \) introduced by Martin(2017). \( \text{SVIX} \) differs from \( \text{VIX} \) if the setting is not conditionally log-normal.
Attention $a_t$ is identical across investors and independent of private signals. Let $a_t$ represent this identical level of attention. It is determined by the price history and satisfies:

$$C'(a_t) = \frac{1}{2} A^{-1} E_t^* [v_{t+\Delta t}].$$

(13)

Asset demand $\theta_{it}$ is the sum of attention weighted private signals and a component $\theta_t$ that is common across investors and only depends on the price history up to the current date $p_0, p_{\Delta t}, \ldots, p_t$:

$$\theta_{it} = \theta_t(p_0, p_{\Delta t}, \ldots, p_t) + A^{-1} \sum_{u=0}^{t-\Delta t} a_u s_{i,u+\Delta t} \Delta t,$$

(14)

Additive separability of signals in asset demand functions is a feature of CARA utility assumption and the normal noise signal structure given in equation (3). Substituting the asset demands into the utility function and simplify, I find that investors face the same attention optimization problem since they have the same risk-aversion and cost of information. Therefore, attention and the risk-neutral variance in equilibrium are identical across investors and independent of the realization of private signals.

Equation (13) establishes a link between investor attention and the risk-neutral variance of the payoff. The right-hand side of this equation represents the marginal value of an additional piece of information to investors. The left-hand side is the marginal cost of information, which is increasing in attention because of the convexity of the cost function $C(a_t)$. When the risk-neutral variance $v_{t+\Delta t}$ is expected to be high, investors are willing to devote more attention to the asset payoff. Intuitively, people acquire more information when they are uncertain about the state of the world.

This risk-neutral variance $v_t$ resembles the volatility index VIX in that they both measure investors’ perceptions of uncertainty in the asset. However, here investors are interested in the variation of the final payoff, as opposed to that of a one-period return defined by the ratio of next period price $p_{t+\Delta t}$ to the current price $p_t$. Falling asset price is usually accompanied by an upward spike in the VIX index because of the leverage effect. This is not necessarily true for the risk-neutral variance of the terminal payoff. Its correlation with the asset price changes with the skewness and support of the payoff distribution and will be further analyzed in Section 4.3.

### 3.2 Market Clearing and State Variables

From market clearing, the aggregate demand for the asset is equal to its supply. Integrate $\theta_{it}$ in (14) over investors. By the law of large numbers, noises in private signals are canceled.
out and the average of $s_{iu}$ is exactly equal to $y$

$$\int_0^1 \left( \theta_I + A^{-1} \sum_{u=0}^{t-\Delta t} a_u s_{t,u+\Delta t} \Delta t \right) di = \theta_I + A^{-1} \left( \sum_{u=0}^{t-\Delta t} a_u \Delta t \right) y = z_t. \quad (15)$$

Investors are infinitely small and the impact of their private signals on the prices is canceled out by the law of large numbers. It is sufficient to characterize the equilibrium using the information publicly available to all investors, which is the history of prices up to the current date $p_0, p_{\Delta t}, \ldots, p_t$.

The demand curve is downward sloping in this model: the common component of asset demand $\theta_I$, implicitly defined in (14), is strictly decreasing in the current price $p_t$. The mapping from $(p_0, p_{\Delta t}, \ldots, p_t)$ to $(\theta_0, \theta_{\Delta t}, \ldots, \theta_t)$ is one-to-one.\(^4\) In other words, the time series $\theta_0, \theta_{\Delta t}, \ldots, \theta_t$ is informationally equivalent to the price history.

At date $t + \Delta t$, the sequence $\theta_0, \theta_{\Delta t}, \ldots, \theta_{t+\Delta t}$ represents all publicly available information. Let us consider $A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$

$$A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) = y + A(a_t \Delta t)^{-1}(z_{t+\Delta t} - z_t). \quad (16)$$

It is a public signal that communicates payoff perturbed by a normal noise with precision $A^{-2} \sigma^2 a_t^2 \Delta t$. This signal represents new public information that arrives at date $t + \Delta t$. The noise in this signal is proportional to the increment in asset supply $z_{t+\Delta t} - z_t$ and is independent of all previous public signals because the supply follows a random walk.

**State Variables**

One difficulty involved in solving this model is that the dimension of state space grows as time increases. $\theta_t$ itself is not Markovian. Both $p_0, p_{\Delta t}, \ldots, p_t$ and $\theta_0, \theta_{\Delta t}, \ldots, \theta_t$ have dimensions equal to the number of trading dates. Fortunately, the information content of these sequences can be summarized into 4 state variables, including the common component of asset demand $\theta_t$ and other 3 state variables that I define below.

**Definition 2.** Expected payoff $m_t$, public information precision $\chi_t$ and private information

\(^4\)I prove this result by mathematical induction. At the starting date, $p_0 \rightarrow \theta_0$ is injective because $\theta_0(p_0)$ is strictly decreasing. Suppose that for date $t - \Delta t$, each set of asset prices $(p_0, p_{\Delta t}, \ldots, p_{t-\Delta t})$ correspond to only one set of demands $(\theta_0, \theta_{\Delta t}, \ldots, \theta_{t-\Delta t})$. Fixing $p_0, p_{\Delta t}, \ldots, p_{t-\Delta t}$, each $p_t$ corresponds to only one $\theta_t$ because $\theta_t(\ldots, p_t)$ is strictly decreasing in $p_t$. This completes the mathematical induction.
precision $\tau_t$ are defined by:

$$m_t \equiv E[y|p_0, p_{\Delta t}, \ldots, p_t]. \quad (17)$$

$$\chi_t \equiv \sum_{u=0}^{t-\Delta t} A^{-2}\sigma_z^2 a_u^2 \Delta t. \quad (18)$$

$$\tau_t \equiv \sum_{u=0}^{t-\Delta t} a_u \Delta t. \quad (19)$$

Investors use the public signals from (16) to update their beliefs about the final payoff. Public information is represented by two state variables, expected payoff $m_t$ and public information precision $\chi_t$. Expected payoff is the expected value of $y$ using only public signals, ignoring all private signals. Public information precision is defined by the aggregate precision of all public signals from date 0 to date $t$. It represents the aggregate amount of information that is publicly available to investors.

Analogously, I use $\tau_t$ to denote the aggregate precision of private signals. It measures the amount of information privately acquired by investors from date 0 to date $t$. By substituting the definition of $\tau_t$ (19) into the market clearing equation (15), I show that the common component of asset demand $\theta_t$ can be expressed as a linear combination of the terminal payoff and the noisy supply:

$$\theta_t = z_t - A^{-1} \tau_t y. \quad (20)$$

The price of and the risk-neutral variance of the terminal payoff can be expressed as functions of these 4 state variables. Price is decreasing in asset demand $\theta_t$ and increasing in $m_t$, which represents investors’ expectation of the terminal payoff from public information. The risk-neutral variance $v_t$ is decreasing in information precisions $\chi_t$ and $\tau_t$, because investors feel less uncertain about the asset payoff if they are more informed.

**State Variable Dynamics**

Information about the final payoff is gradually incorporated into private and public signals. The amount of public and private information increases and investors update their beliefs about the expected payoff. Lemma 2 summarizes how these state variables evolve from date $t$ to $t + \Delta t$. 

12
Lemma 2. The dynamics of state variables \( \theta_t, m_t, \chi_t \) and \( \tau_t \) are given by:

\[
\begin{align*}
\theta_{t+\Delta t} &= \theta_t - A^{-1} a_t y \Delta t + z_{t+\Delta t} - z_t, \\
m_{t+\Delta t} &= m_t + A^{-2} \sigma_z^{-2} a_t^2 h_t (y - m_t) \Delta t - A^{-1} \sigma_z^{-2} a_t h_t (z_{t+\Delta t} - z_t) + o(\Delta t), \\
\chi_{t+\Delta t} &= \chi_t + A^{-2} \sigma_z^{-2} a_t^2 \Delta t, \\
\tau_{t+\Delta t} &= \tau_t + a_t \Delta t.
\end{align*}
\]

where

\[
h_t = \text{Var}[y|p_0, p_{\Delta t}, \ldots, p_t]
\]

is a deterministic function of \( m_t \) and \( \chi_t \) and represents the variance of \( y \) in the objective physical measure after observing public information \( p_0, p_{\Delta t}, \ldots, p_t \).

The aggregate amount of private information is increasing at a speed of attention, as expressed in the equation (24). The amount of public information is growing at a speed proportional to the square of attention, which is also the rate at which the expected payoff \( m_t \) drifts towards the direction of its true value \( y \). A higher level of attention indicates that information is both acquired and disseminated at a quicker pace.

3.3 Equations for Price and Risk-neutral Variance

In the following section, I derive a system of recursive equations for price \( p_t \) and the risk-neutral variance \( v_t \). Because the interest rate is zero, the price is a martingale under the risk-neutral measure:

\[
p_t = E_t^*(p_{t+\Delta t}).
\]

Using the law of total variance, I decompose \( v_t \) as the sum of expected next-period variance and variance of next-period expectation:

\[
v_t = \text{Var}_t^*(y) = E_t^*[\text{Var}_{t+\Delta t}^*(y)] + \text{Var}_t^*[E_{t+\Delta t}^*(y)]
\]

\[
= E_t^*[v_{t+\Delta t}] + \text{Var}_t^*(p_{t+\Delta t}).
\]

The stochastic discount factor from date \( t \) to date \( T \) is required to change the probability measure from the risk-neutral one to the objective one. Lemma 3 specifies this stochastic discount factor \( \xi_{t,T} \).

Lemma 3. Let \( \xi_{t,T}^i \) denote the ratio of investor i’s marginal utility \( U'(W_{iT}) \) and its date \( t \) conditional expectation \( E_t^i[U'(W_{iT})] \):

\[
\xi_{t,T}^i = \frac{U'(W_{iT})}{E_t^i[U'(W_{iT})]}.
\]
Let \( \xi_{t,T} \) represent the average of \( \xi^i_{t,T} \) across investors. It is a valid SDF and is given by:

\[
\xi_{t,T} = E \left[ \xi_{t,T} \mid p_0, p_{\Delta t}, \ldots, p_{T - \Delta t}, y \right] = \exp \left( - \sum_{u=t}^{T-\Delta t} \left[ A \theta_u (p_{u+\Delta t} - p_u) + \frac{1}{2} \tau_u (p_{u+\Delta t}^2 - p_u^2) - A \cdot C(a_u) \Delta t \right] - f_t \right),
\]

(29)

where \( f_t \) is a normalizing variable defined by:

\[
f_t = \ln E_t \left[ \exp \left( - \sum_{u=t}^{T-\Delta t} \left[ A \theta_u (p_{u+\Delta t} - p_u) + \frac{1}{2} \tau_u (p_{u+\Delta t}^2 - p_u^2) - A \cdot C(a_u) \Delta t \right] \right) \right].
\]

(31)

such that \( E_t(\xi_{t,T}) = 1 \).

\( f_t \) is an auxiliary variable that helps form a system of equations involving the price and the risk-neutral variance. Substituting (29) into (27) and (26), use law of iterated expectations, and simplify, I arrive at the following recurrence equations for \( p_t \) and \( v_t \):

\[
p_t = E_t \left[ \exp \left( -A \theta_t (p_{t+\Delta t} - p_t) - \frac{1}{2} \tau_t (p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t + f_t \right) \right] p_{t+\Delta t},
\]

\[
v_t = E_t \left[ \exp \left( -A \theta_t (p_{t+\Delta t} - p_t) - \frac{1}{2} \tau_t (p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t + f_t \right) \right] v_{t+\Delta t} + \text{Var}_t^1(p_{t+\Delta t}).
\]

(32)

Definition (31) can also be rewritten recursively:

\[
\exp(f_t) = E_t \left[ \exp \left( -A \theta_t (p_{t+\Delta t} - p_t) - \frac{1}{2} \tau_t (p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t + f_{t+\Delta t} \right) \right].
\]

(33)

Equilibrium in the Continuous-time Limit

A continuous-time limit approach confers several advantages over approaching (32) and (33) directly in the discrete-time setup. It is challenging to express \( E^*_t[v_{t+\Delta t}] \) in (13) as a function of date \( t \) state variables. This issue is sidestepped by taking the limit \( \Delta t \to 0 \), in which case \( E^*_t[v_{t+\Delta t}] \to v_t \).

In addition, these integral equations simplify to partial differential equations in continuous time. Let \( \mu_{pt} \) and \( \sigma_{pt} \) denote the instantaneous drift and volatility of price \( p_t \) in the continuous-time limit:

\[
\mu_{pt} = \lim_{\Delta t \to 0} \frac{E_t[p_{t+\Delta t} - p_t]}{\Delta t}, \quad \sigma^2_{pt} = \lim_{\Delta t \to 0} \frac{E_t[(p_{t+\Delta t} - p_t)^2]}{\Delta t}.
\]

(34)

Applying the Taylor series expansion to \( p(t, \theta_t, m_t, \chi_t, \tau_t) \) and substituting in the state
variable dynamics from Lemma 1, drift $\mu_{pt}$ and volatility $\sigma_{pt}$ are given by:

\[ \mu_{pt} = \frac{\partial p}{\partial t} - \frac{\partial p}{\partial \theta} A^{-1} a_t m_t + \frac{\partial p}{\partial \chi} A^{-2} \sigma^2 z_t^2 a_t^2 + \frac{\partial p}{\partial \tau} a_t + \frac{1}{2} \frac{\partial^2 p}{\partial \theta^2} \sigma^2, \]

\[ + \frac{1}{2} \frac{\partial^2 p}{\partial m^2} (A^{-1} \sigma^{-1} a_t h_t)^2 - \frac{\partial^2 p}{\partial \theta \partial m} A^{-1} a_t h_t \]

\[ \sigma_{pt} = \frac{\partial p}{\partial m} A^{-1} \sigma^{-1} a_t h_t - \frac{\partial p}{\partial \theta} \sigma. \]

(35)

(36)

$\mu_{vt}$, $\sigma_{vt}$, $\mu_{ft}$ and $\sigma_{ft}$, the drifts and volatilities of variables $v_t$ and $f_t$, are similarly defined by replacing $p$ with $v$ or $f$ in (34) and (36).

Proposition 2 describes this system of partial differential equations. The terminal conditions are reported in the appendix.

**Proposition 2.** Price, risk-neutral variance, and the normalizing variable in stochastic discount factor, as functions of time and state variables $\theta_t$, $m_t$, $\chi_t$ and $\tau_t$ satisfy:

\[ \mu_{pt} + \sigma_{pt} \sigma_{ft} - (A \theta_t + \tau p_t) \sigma^2_{pt} = 0, \]

(37)

\[ \mu_{vt} + \sigma_{vt} \sigma_{ft} - (A \theta_t + \tau p_t) \sigma_{vt} \sigma_{pt} + \sigma^2_{pt} = 0, \]

(38)

\[ \mu_{ft} + \frac{1}{2} (\sigma_{ft})^2 - \frac{1}{2} (A \theta_t + \tau p_t)^2 \sigma^2_{pt} + A \cdot C(a_t) = 0. \]

(39)

where attention is implicitly determined by:

\[ C'(a_t) = \frac{1}{2} A^{-1} v_t. \]

(40)

This system of equations is solved numerically by the finite-difference method on a five-dimensional grid of time $t$ and four state variables $\theta_t$, $m_t$, $\chi_t$, and $\tau_t$. A noteworthy by-product of this analysis is the drift in price dynamics $\mu_{pt}$, which also represents the instantaneous expected return $E_t[dp_t]/dt$. The solution to price, risk-neutral variance, attention, and expected return facilitates the analysis of this equilibrium.

### 4. Asset Pricing Implications

Different realizations of the exogenous noisy supply of the asset leads to different endogenous dynamics of price, uncertainty and information acquisition. The price is decreasing in the noisy supply of the asset. When the payoff distribution is not normal, uncertainty measured by the risk-neutral variance also varies with the noisy supply. If uncertainty is high (compared to other realizations of exogenous shocks), investors acquire more information.
In this section, I investigate how uncertainty, expected return, and past prices are endogenously related through investor attention. Section 4.1 studies the interplay between information acquisition and uncertainty. In Section 4.2, I explore the link between information acquisition and the future expected return. Section 4.3 relates cyclicality of attention to skewness in the payoff distribution. Findings and predictions in this section are also valid if multiple assets are traded.

4.1 Endogenous Uncertainty Dynamics

Suppose that the economy enters a period of high uncertainty about the asset payoff. High uncertainty induces investors to acquire more information. This causes asset returns to be even more volatile in the early stages of the high-uncertainty period, as agents learn more information. Returns become less volatile in the later stages, as learning gradually reduces payoff uncertainty.

Proposition 1 establishes the link between uncertainty and information acquisition. The risk-neutral variance of the final payoff $\nu_t$ measures uncertainty and represents the marginal value of an additional piece of information to investors. With a high level of uncertainty, information becomes valuable, and investors accumulate it at a quicker pace through a higher level of attention $a_t$. The instantaneous volatility of the asset return $\sigma_{pt}$ also rises as a result.

$$\frac{\partial \sigma_{pt}}{\partial a_t} = \frac{\partial p}{\partial m} A^{-1} \sigma_z^{-1} > 0$$ (41)

When investors acquire more precise private signals, they also make the asset demand and thus the asset price more informative with regard to the final payoff, which increases the amount of public information available to investors. The dynamics of private and public information precisions in the continuous-time limit are as follows:

$$\tau' = a_t, \quad \chi' = A^{-2} \sigma_z^2 a_t^2.$$ (42)

Private information accumulates at the speed of attention, and public information accumulates at a speed proportional to the square of attention. Information reduces investors’ perception of uncertainty in the asset. A higher level of attention indicates that this reduction of uncertainty happens more quickly. Let $\mu_{vt}^*$ denote the instantaneous drift of $\nu_t$ in the risk-neutral measure:

$$\mu_{vt}^* = \lim_{\Delta t \to 0} \frac{E_t^*[\nu_{t+\Delta t} - \nu_t]}{\Delta t}.$$ (43)
Proposition 3 reports the expression of this expected change in risk-neutral variance.

**Proposition 3.** The risk-neutral drift of risk-neutral variance $v_t$ is given by:

$$\mu^*_v t = -\left(\frac{\partial p}{\partial \theta}\right)^2 \sigma^2_z + 2 \frac{\partial p}{\partial \theta} \frac{\partial p}{\partial m} A^{-1} a_t h_t - \left(\frac{\partial p}{\partial m}\right)^2 A^{-2} \sigma^2_z a_t^2 h_t^2.$$  \hspace{1cm} (44)

It is decreasing in investor attention:

$$\frac{\partial \mu^*_v t}{\partial a_t} < 0.$$  \hspace{1cm} (45)

The last term in equation (44) represents the contribution of public information to the reduction of uncertainty. It is decreasing in attention and contains $A^{-2} \sigma^2_z a_t^2$, the speed at which public information disseminates. This effect is prominent at high levels of uncertainty because it is proportional to attention squared. The arrival of public information updates investors' belief and makes the distribution of the final payoff more concentrated around its mean, which contributes to the decline of the risk-neutral variance $v_t$.

The second term in (44) is also a decreasing function of attention. A higher expected payoff $m_t$ shifts the risk-neutral distribution of the final payoff to the right. As a result, the current price $p_t$ is increasing in $m_t$. Similarly, the price is decreasing in the common component of asset demand $\theta_t$:

$$\frac{\partial p}{\partial m} > 0, \quad \frac{\partial p}{\partial \theta} < 0.$$  \hspace{1cm} (46)

The drift of uncertainty in the objective physical measure $\mu_v t$ differs from $\mu^*_v t$ by the variance risk premium. Depending on the shape and skewness of the risk-neutral distribution, it either reinforces or diminishes the positive correlation between attention and the reduction in uncertainty. However, for a realistic choice of parameters this channel is unlikely to overtake the direct effect of information dissemination analyzed above.

When uncertainty is high, investor attention increases and reduction in uncertainty happens more quickly. An upward spike of the risk-neutral variance is usually followed by a rapid reduction. As uncertainty decreases, the return of the asset also becomes less volatile. This is consistent with the empirical observation that peaks in VIX are usually followed by a rapid decline.\footnote{Harvey and Whaley (1992) demonstrate that changes in implied volatilities are negatively predicted by the lagged changes and that the explanatory power is higher in the sample that includes the crash. Mencia and Sentana (2013) suggest that during the 2008-2009 crisis volatility exhibits more mean reversion than that in the past.}

Figure 1 illustrates 2 sample paths of the risk-neutral variance.
Figure 1: Sample Paths of the Risk-Neutral Variance

The figure demonstrates 2 sample paths of the risk-neutral variance $v_t$ simulated from the model. The parameters are set as follows: $T = 3$, $y \sim \text{Lognormal}(0, \sigma_y^2)$ where $\sigma_y = 0.702$, $z_0 = 5$, $\sigma_z = 0.15$, $A = 0.2$, and $C(a_{it}) = 5(a_{it})^2$.

The dynamics of uncertainty and risk-neutral variance also have a profound impact on the dynamics of information acquisition. Each piece of information diminishes the value of the following piece. High attention levels make subsequent information acquisition less profitable and are therefore unlikely to be sustained.

4.2 Expected Return and Attention

The expected return of an asset is positively associated with investor attention. When attention is high today, investors acquire information at a rapid pace, leading to a rapid reduction in uncertainty. Then, the asset becomes less risky tomorrow and enjoys a lower risk premium tomorrow and beyond. This quick reduction in tomorrow’s risk premium compared to today’s corresponds to a high expected return for the asset.

The asset’s expected payoff is $m_t$. The expectation of payoff in the risk-neutral measure is the price $p_t$. Their difference $m_t - p_t$ is this asset’s risk premium. Investors’ acquisition and dissemination of information make the asset less risky and shrink its risk premium towards zero.
Let us move back to the discrete-time setup for a moment. The risk premium at date \(t+\Delta t\) is \(m_{t+\Delta t} - p_{t+\Delta t}\). Because the expected payoff is a martingale \(m_t = E_t[\mathcal{B}^{m}_{t+\Delta t}]\), the expected return from date \(t\) to \(t+\Delta t\) is equal to the expected reduction of risk premium in the same period:

\[
E_t[p_{t+\Delta t} - p_t] = E_t\left[ \frac{m_t - p_t}{\text{date } t \text{ risk premium}} - \frac{m_{t+\Delta t} - p_{t+\Delta t}}{\text{date } t + \Delta t \text{ risk premium}} \right]. \tag{47}
\]

When the asset is trading at a price \(p_t\) below the expected payoff \(m_t\), its risk premium is positive and shrinks towards zero from above. High attention implies a quick reduction in risk premium and hence a high expected return. Proposition 4 reports the expected return in the continuous-time limit and relates it to attention and risk premium.

**Proposition 4.** The instantaneous expected return of the asset \(\mu_{pt}\) is given by:

\[
\mu_{pt} = \left[ \frac{(m_t - p_t)}{\text{risk premium}} A^{-1} \sigma_z^{-1} a_t + \sigma_z E_t \left[ \int_t^T \frac{\partial^2 p_u}{\partial \theta_u^2} A \sigma^2 u du \right] + A^{-1} \sigma_z^{-1} h_t a_t E_t \left[ \int_t^T \frac{\chi u}{\chi_t} \frac{\partial^2 p_u}{\partial m_u^2} a_u h_u du \right] \right] \cdot \left( \frac{\partial p_t}{\partial m_t} A^{-1} \sigma_z^{-1} h_t a_t + \left( \frac{\partial p_t}{\partial \theta_t} \sigma_z \right) \right). \tag{48}
\]

If the risk premium \(m_t - p_t\) is greater than the second-order derivatives terms in the above expression, the expected return \(\mu_{pt}\) is increasing in attention \(a_t\).

The expected return is positively associated with investor attention as long as the risk premium is not exceptionally low, or the curvature of asset demand is not exceptionally negative. The second-order derivatives \(\partial^2 p_u/\partial \theta_u^2\) and \(\partial^2 p_u/\partial \theta_u^2\) appear because of the non-normality in the payoff distribution.

This prediction provides an alternative explanation to a list of well-documented empirical regularities. Da, Engelberg and Gao (2011) use Google Search Volume Index to measure investor attention directly. They find that an increase in search frequency predicts higher stock prices in the following two weeks. Lou (2014) documents that advertising can attract investor attention and impact stock returns in the short run: an increase in advertising spending is accompanied by a contemporaneous rise in retail buying and higher abnormal stock returns. Lee and So (2017) show that analyst coverage predicts stock return: firms with abnormally high analyst coverage subsequently outperform firms with abnormally low coverage by approximately 80 basis points per month.

The above analysis suggests that investor attention predicts the asset return. Because attention is determined by certainty, the model also implies that the risk-neutral variance...
Figure 2: Uncertainty and Expected Return of the Asset
The figure shows the autocorrelation between the risk-neutral variance in panel (a) and the correlation between present risk-neutral variance \( v_t \) and future expected return \( \mu_{p,t+u} \) in panel (b). The current date is \( t = 1 \). Other parameters are set as follows: \( T = 3, y \sim \text{Lognormal}(0, \sigma_y^2) \) where \( \sigma_y = 0.702, z_0 = 5, \sigma_z = 0.15, A = 0.2, \) and \( C(a_{it}) = 5(a_{it})^2 \).

\( v_t \) positively forecasts the expected return at the same date. \( v_t \) is also useful at predicting \( \mu_{p,t+u} \), the expected return at a future date \( t + u \), which changes with attention and uncertainty at that date. Since information is acquired and diffused only gradually through the investing public, date \( t + u \) risk-neutral variance \( v_{t+u} \) is positively correlated to the date \( t \) measure. This translates into a correlation between \( v_t \) and \( \mu_{p,t+u} \), illustrated in the right panel of Figure 2. This prediction is also supported by empirical evidence. Martin (2017) find that the risk-neutral variance predicts the return of the market at horizons from one month to one year.

However, this correlation becomes weaker as the time interval \( u \) increases, because the autocorrelation of the risk-neutral variance decreases over time. High uncertainty implies increased attention and information acquisition, which in turn drives down uncertainty. Consequently, the correlation between \( v_t \) and \( v_{t+u} \) decreases as the time interval \( u \) increases from 1 month to 24 months, as is shown in the left panel of Figure 2.

4.3 Cyclicality of Information Acquisition
The cyclicality of investor attention depends on the skewness of the payoff distribution. When the distribution is positively skewed and bounded from below, investors are more excited about potential upside gains. Therefore they acquire more information when the
The figure demonstrates how asset price changes the shape of the payoff’s risk-neutral distribution. The solid line represents the probability density, and the dashed line correspond to the asset price, which is identical to the mean of this distribution. The prior distribution of the payoff is $y \sim \text{Lognormal}(0, \sigma_y^2)$ where $\sigma_y = 0.702$.

price is high. Conversely, when the distribution is negatively skewed, investors are more worried about potential losses in the downside. They devote more attention to the asset when the price is low.

First, let us consider an individual stock where the payoff distribution $G(y)$ has a lower bound and a fat right tail. At date $t$, investors observe the entire price history up to this date and use this information to update their beliefs about the risk-neutral distribution of payoff. Figure 3 demonstrates how the risk-neutral distribution depends on price in this example.

A low price implies that this probability distribution is concentrated near the lower bound. Conversely, a high price indicates that the payoff is more likely to be in a region with high variation and, thus, more uncertain from investors’ perspective. This establishes a positive correlation between price and the risk-neutral variance. Therefore, attention is procyclical, suggesting that more information is acquired and disseminated in good times than in bad times.

Now, consider a second example where the asset is a bond that pays 0 if it defaults and 1 if not. The risk-neutral distribution of payoff is completely characterized by the bond price $p_t$, which represents the risk-neutral probability that it does not default. The
risk-neutral variance is:

\[ v_t = E_t^*[(y - p_t)^2] = p_t(1 - p_t). \] (49)

If the risk-neutral probability of default does not exceed 1/2, \( p_t \) is greater than 1/2 and in this region, the risk-neutral variance is decreasing in price. When the bond is trading at a price near its face value 1, investors do not acquire much information because the value of an additional signal is close to zero. Any negative news that leads to a decline in price increases the value of information and attracts more attention. Thus information acquisition is countercyclical when \( p_t \in (1/2, 1) \).

For general payoff distributions, the cyclicality of information acquisition is determined by \( \partial v/\partial m \), the sensitivity of the risk-neutral variance to the expected payoff. Attention is determined by uncertainty and the price is increasing in the expected payoff \( m_t \). Therefore, attention is procyclical if \( \partial v/\partial m \) is positive and countercyclical if it is negative. Proposition 5 discusses how uncertainty moves in line with the price at date \( T - \Delta t \), the last trading date before the asset realizes its payoff.

**Proposition 5.** At date \( T - \Delta t \), uncertainty is increasing in price if the risk-neutral distribution is positively skewed. The sensitivity of the risk-neutral variance \( v_t \) to the expected payoff \( m_t \) is given by:

\[ \frac{\partial v}{\partial m} = \frac{1}{h_{T-\Delta t}}E_{T-\Delta t}[(y - p_{T-\Delta t})^3]. \] (50)

\( \partial v/\partial m \) has the same sign as the risk-neutral skewness of the payoff. If the distribution is right-skewed, uncertainty is increasing in the expected payoff and so is the price. The converse is true if the distribution is left-skewed. Unfortunately, \( \partial v/\partial m \) does not admit a simple expression at dates other than \( T - \Delta t \). It is not only influenced by the risk-neutral skewness but also relies on the interaction between distribution and other state variables.

The model predicts that information acquisition is procyclical for firm-specific information and countercyclical for market-wide information, consistent with empirical evidence. Idiosyncratic components of stock returns and payoffs tend to be right-skewed\(^6\), implying that stock-specific information is more valuable in good times. Hong, Lim and Stein (2000) find that negative firm-specific information diffuses more slowly compared to positive news. The market return, on the contrary, is left-skewed. Garcia (2013) finds that in

\(^6\)Kothari and Warner (1997) document that abnormal returns estimated using four models (market model, market-adjusted model, capital asset pricing model, and Fama French three-factor model) are all positively skewed.
times of hardship investors react strongly to business cycle news, while in good times the predictability of media content on Dow Jones Industrial Average is much weaker.

5. **Extension**

I consider a simple multi-asset extension of the model that explains momentum in the stock market. In this economy, a risk-free asset, several individual stocks, and the market are traded. Each individual stock’s final payoff is the sum of a stock-specific idiosyncratic component and a market component identical for all stocks. The focus of analysis in this section is information acquisition with respect to this idiosyncratic component of the payoff. It determines the return of a stock in excess of the market and explains why it is positively autocorrelated.

5.1 **Setup**

Investors trade 1 risk-free asset, \( n \) individual stocks, and 1 asset representing the market. Stock \( j \in \{1, 2, \ldots, n\} \) has final payoff \( y_0 + y_j \), which consists of a market component \( y_0 \) and a stock-specific idiosyncratic component \( y_j \) that is independent of the market and across stocks. The market payoff itself is also traded and there is a market asset that pays \( y_0 \) at the final date.

Payoffs \( y_0 \) and \( y_j \) are unobservable to investors at the start and they are distributed with cumulative distribution functions \( G_0(y_0) \) and \( G_i(y_j) \). The distribution of stock-specific component \( y_j \), \( G_j(y_j) \), is bounded from below and positively skewed. A number of factors contribute to this asymmetry in the stock payoff. Limited liability for equity holders indicates that the investments in stocks have bounded downside risk but some potential for a large upside gain. Besides, the firm may possess a real option to expand the business when it is doing well which further boosts the upside potential.

Let \( p_{0t} \) denote the date \( t \) price of the market asset and \( p_{jt} \) denote the price of a claim that pays \( y_j \) at the final date. The price of individual stock \( j \) which pays \( y_0 + y_j \) is \( p_{0t} + p_{jt} \). \( p_{jt} \) represents the price of the idiosyncratic payoff component, and it is equal to the difference in price between stock \( j \) and the market asset. For the period between date \( t \) and \( t + \Delta t \), the return of the market asset is \( p_{0,t+\Delta t} - p_{0t} \) and the return of stock \( j \) in excess of the market is:

\[
[(p_{0,t+\Delta t} + p_{j,t+\Delta t}) - (p_{0t} + p_{jt})] - [p_{0,t+\Delta t} - p_{0t}] = p_{j,t+\Delta t} - p_{jt}. \tag{51}
\]
Let $z_{jt}$ represent the supply of stock $j$ and $z_{0t}$ represent the aggregate supply of all risky assets, including the market asset and all $n$ individual stocks. Since the payoff of any risky asset contains a market payoff component, $z_{0t}$ also represents the supply of this component $y_0$ in the economy. Similar to Kacperczyk, Van Nieuwerburgh and Veldkamp (2016), increments in the supplies of payoff components $z_{0,t+\Delta t} - z_{0t}$ and $z_{jt,t+\Delta t} - z_{jt}$ are assumed to be independent. As in the baseline model, they both follow random walks and have variances $\sigma_{z0}^2 \Delta t$ and $\sigma_{zj}^2 \Delta t$ respectively:

$$z_{0,t+\Delta t} - z_{0t} \sim i.i.d. \mathcal{N}(0, \sigma_{z0}^2 \Delta t),$$
$$z_{j,t+\Delta t} - z_{jt} \sim i.i.d. \mathcal{N}(0, \sigma_{zj}^2 \Delta t), \quad j = 1, 2, \ldots, n. \quad (52)$$

The cost of information acquisition is additive across different payoffs. Let $a_{it}^j$ represent investor $i$’s attention towards payoff $y_j$. The aggregate cost of information for investor $j$ from date $t$ to date $t + \Delta t$ is the sum of that for each payoff $\sum_{j=0}^{n} C(a_{it}^j) \Delta t$. Contrary to standard rational inattention models (e.g., Sims (2003)), which impose a fixed capacity upper bound for the aggregate attention on all assets, I assume that information acquisition is independent across payoffs and different assets do not compete for investor attention. Increased attention on one payoff does not raise the cost of information for other payoffs. Investors acquire less information in aggregate when the market is devoid of profitable investment opportunities.

The above assumptions about asset supply and information acquisition simplify the analysis of equilibrium. The portfolio choice for different payoffs can be solved separately because payoffs $y_0, y_1, \ldots, y_n$ are independent and investors have CARA preference. Since the cost of information is additive across assets, attention choice can also be solved separately. The characterization of equilibrium is therefore identical to the single asset baseline model, with $\theta_{jt}, m_{jt}, \chi_{jt}$ and $\tau_{jt}$ replacing $\theta_t, m_t, \chi_t$ and $\tau_t$ as state variables.

5.2 Analysis of Equilibrium

I consider a numerical example with parameters provided in Table 1.\footnote{The analysis here focuses on idiosyncratic payoffs of individual stocks and does not extend to the market. In section 5.2, $j$ to refers one of 1, 2, \ldots, $n$ and does not include 0.} The final payoff is realized in $T = 3$ years. For simplicity, I assume the idiosyncratic payoff components for different stocks have the same distribution $y_j \sim \text{Lognormal}(0, \sigma_y^2)$, where $\sigma_y$ is set at 0.702 to match the skewness of 36-month buy-and-hold CAPM abnormal return at a level
Table 1: Parameter Values

The table lists values assigned to the length of investment $T$; the distribution of idiosyncratic payoff $G_j(y_j)$ and parameter $\sigma_y$; the starting supply of individual stock $z_{j0}$; the volatility of stock supply $\sigma_{zj}$; the absolute risk aversion parameter $A$; and the cost of information $C(a_{it}^j)$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$G_j(y_j)$</th>
<th>$\sigma_y$</th>
<th>$z_{j0}$</th>
<th>$\sigma_{zj}$</th>
<th>$A$</th>
<th>$C(a_{it}^j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Lognormal($0, \sigma_y^2$)</td>
<td>0.702</td>
<td>5</td>
<td>0.15</td>
<td>0.2</td>
<td>5($a_{it}^j$)$^2$</td>
</tr>
</tbody>
</table>

The starting supply of individual stock is $z_{j0} = 5$, and the volatility of stock supply is set at $\sigma_{zj} = 0.15$, such that daily trading volume is approximately 0.2 percent of the outstanding shares. The absolute risk aversion is $A = 0.2$, which corresponds to relative risk aversion around 10 at the initial date for an investor with wealth equal to 10 times its investment in one stock. Last, the cost of information is assumed to be a quadratic function of attention $C(a_{it}^j) = 5(a_{it}^j)^2$.

Attention and Past Excess Return

For the period between date 0 and date $t$, the return of stock $j$ in excess of the market is $p_{jt} - p_{j0}$. In equilibrium this return is determined by state variables $\theta_{jt}$, $m_{jt}$, $\chi_{jt}$ and $\tau_{jt}$. In particular, it is increasing in the expected payoff $m_{jt}$, which summarizes public information about this stock’s idiosyncratic payoff. Attention to this stock-specific payoff tends to be procyclical because its distribution is bounded from below and skewed to the right.

In this numerical example, the risk-neutral variance of idiosyncratic payoff $v_{jt}$ is indeed increasing in its expected value. Figure 4 illustrates the relationship between $v_{jt}$ and $m_{jt}$ at date $t = 1$, one year after the starting date. The other 3 state variables $\theta_{jt}$, $\chi_{jt}$ and $\tau_{jt}$ are fixed at the median of their distributions.

Excess return and risk-neutral variance are both increasing in the expected payoff. This contributes to a positive correlation between a stock’s past performance in excess of the market and investors’ current attention to its idiosyncratic payoff. High excess return in the past indicates that this payoff is likely to be in a range with high variation, and thus it is more valuable for investors to focus on this piece of information. Consider a

\[8\text{Kothari and Warner (1997) find that 36-month buy-and-hold abnormal return with respect to CAPM has a skewness of 2.90. Other moments including kurtosis and quartiles are also similar to those implied by a log-normal distribution.}\]
The figure plots the risk-neutral variance $v_{jt}$ as a function of the expected payoff $m_{jt}$ for the idiosyncratic payoff component of a stock. Other state variables are fixed at the middle of their distribution. The parameters are set in Table 1.

The left panel of Figure 5 shows a scatter plot for excess return in the first year $p_{j1} - p_{j0}$ and attention at the end of the first year for 20,000 simulated time-series.

**Expected Excess Return and Attention**

In the baseline model, Section 4.2 establishes that the expected return is increasing in attention if the asset enjoys a positive risk premium. Similar results hold true for the multi-asset case. High attention to the stock’s idiosyncratic payoff implies a quick reduction in the risk-premium concerning this payoff, which in turn contributes to high excess return for the stock. The right panel of Figure 5 demonstrates the end of the first year attention $a_t$ and instantaneous expected excess return $\mu_{jt}$ for simulated data.

**Serial Correlation of Excess Return**

Investor attention to a stock is positively correlated with its past excess return. The expected excess return of this stock in the future increases with attention. Combining these two results, I find that excess return of a stock exhibits positive autocorrelation because of endogenous information acquisition.

The average return of a sufficiently large group of individual stocks approximates that of the market. Therefore, a stock’s excess return is high if and only if it performed well
Figure 5: attention and Excess Return

The figure plots past excess return $p_{jt} - p_{j0}$, attention $a_{jt}$ and the instantaneous expected excess return $\mu_{jt}$ at year $t = 1$ for 20,000 simulated time-series. Panel (a) shows that attention is increasing in past excess return, and Panel (b) illustrates that expected excess return is increasing in attention. The parameters are set in Table 1.

6. Concluding Remarks

In this paper, I developed a noisy rational expectations model with endogenous information acquisition and used it to analyze the joint dynamics of attention, price, and uncertainty. The starting point of this analysis is equation (13), which shows that investor attention is determined by the uncertainty measure risk-neutral variance. It is empirically measurable from option prices and resembles the volatility index VIX. Conversely, attention determines investment choices and thus affects the dynamics of asset prices and of uncertainty. This
The figure shows the estimated value of the regression coefficient $\beta_{j}^{t}$ for the equation $\mu_{pt} = \alpha_{t+u} + \beta_{t+u} (p_{jt} - p_{j0}) + e_{t+u}$ from simulated data. The parameters are set in Table 1.

Figure 6: Serial Correlation of Excess Return

 interaction between attention and uncertainty creates rich asset pricing dynamics.

This model generates predictions that are qualitatively different from those in static and normal distribution models. First, high uncertainty attracts more attention, which in turn reduces both uncertainty and attention. Episodes of high uncertainty and attention are therefore unlikely to be sustained. Second, information acquisition drives down both uncertainty and risk premium. The expected return, which is identical to the expected reduction of risk premium, increases with investor attention. Third, the correlation between price and the risk-neutral variance depends on the skewness and support of the payoff distribution. Information acquisition tends to be procyclical for right-skewed payoffs and countercyclical for left-skewed ones. These predictions are consistent with empirical observations.

In the extension, I applied the above results to a multi-asset setup and illustrated that past winners tend to continue to perform well relative to the market. The idiosyncratic component of stock payoff is right-skewed because of limited liability and real option to expand. Stocks that performed well relative to the market have high uncertainty and attract more attention and, hence, are expected to continue to generate high excess returns. Because the dynamics of uncertainty contain a mean-reverting component, the serial correlation of excess return weakens as the horizon increases.
References


Appendix

Proof of Lemma 1.

Optimization (8) is equivalent to:

$$\max_{\theta_i} \int - \exp \left( -A\left( \theta_i - A^{-1}a_{i0} s_{i1} \right) (y - p_1) - \frac{a_{i0}}{2} (y^2 - p_1^2) + A \cdot C(a_{i0}) \right) \cdot \frac{1}{\sqrt{2\pi(a_{i0})^{-1}}} \exp \left( -\frac{a_{i0}}{2} (s_{i1} - p_1)^2 \right) dG(y|p_0, p_1).$$  \hfill (A1)

$A \cdot C(a_{i0})$ and $\exp \left( -a_{i0}/2 \cdot (s_{i1} - p_1)^2 \right)$ only contain known information at date 1 and could be taken out of the expression. Choosing optimal $\theta_i$ is equivalent to choosing optimal $\theta_i - A^{-1}a_{i0} s_{i1}$:

$$\max_{\theta_i - A^{-1}a_{i0} s_{i1}} \int - \exp \left( -A\left( \theta_i - A^{-1}a_{i0} s_{i1} \right) (y - p_1) - \frac{a_{i0}}{2} (y^2 - p_1^2) \right) \cdot (y - p_1) \cdot dG(y|p_0, p_1) = 0. \hfill (A2)$$

This optimization problem is identical for different investors and concave in $\theta_i - A^{-1}a_{i0} s_{i1}$. Take the first-order condition and simplify:

$$\int - \exp \left( -A\left( \theta_i - A^{-1}a_{i0} s_{i1} \right) (y - p_1) - \frac{a_{i0}}{2} (y^2 - p_1^2) \right) \cdot (y - p_1) \cdot dG(y|p_0, p_1) = 0. \hfill (A3)$$

Because the moment generating function always exists, $G(y)$ has exponentially bounded tails. For price $p_1$ that belongs to the support of $G(y)$, the solution to (A3) exists and is unique. As a result, $\theta_i - A^{-1}a_{i0} s_{i1}$ must be the same across investors. Define

$$\theta_1 = \theta_i - A^{-1}a_{i0} s_{i1}. \hfill (A4)$$

$\theta_1$ represents a component of asset demand that is identical across investors. $\theta_1$ is a function of prices $p_0$ and $p_1$ because (A2) relies on $dG(y|p_0, p_1)$. Identity (A4) is equivalent to (9).

Proof of Proposition 1.

I prove this proposition by backward induction. Suppose asset demand equation (14) holds for dates $t = t_0 + \Delta t, \ldots, T - \Delta t$ and attention equation (13) holds for dates $t = t_0, \ldots, T - \Delta t$.

First I prove that (14) is valid for date $t = t_0$.

Let $p^{t_0}$ denote future prices $p_{t_0 + \Delta t}, \ldots, p_{T - \Delta t}$, and $s^{i0}$ denote investor $i$’s future signals $s_{i,t_0 + \Delta t}, \ldots, s_{i,T - \Delta t}$. The joint distribution of future prices and signals conditional
on all public and private information possessed by investor $i$ at date $t_0$ is represented by $G(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}, s_{i,t_0}, \ldots, s_{i,t_0})$. I use $\pi(s_{i,t_0}, \ldots, s_{i,t_0})$ to denote the probability density function of the signals.

At date $t_0$, investor $i$’s utility is:

$$
E^i_{t_0}[-\exp(-AW_{iT})]
= \int -\exp(-AW_{iT}) dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}, s_{i,t_0}, \ldots, s_{i,t_0})
= \int -\exp(-AW_{iT}) \frac{\pi(s_{i,t}, \ldots, s_{i,t_0}|y)}{\pi(s_{i,t}, \ldots, s_{i,t_0}|p_0, \ldots, p_{t_0})} dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0})
= \int -\exp \left(-AW_{i,t_0} - A \sum_{t=t_0}^{T-\Delta t} \theta_{it}(p_{i,t+\Delta t} - p_{it}) + A \sum_{t=t_0}^{T-\Delta t} C(a_{it})\Delta t \right)
\cdot \prod_{t=0}^{t_0-\Delta t} \frac{1}{\sqrt{2\pi(a_{it}\Delta t)^{-1}}} \exp \left(-\frac{a_{it}\Delta t}{2}(s_{i,t+\Delta t} - p_{it})^2 \right) dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}).
$$

(A5)

In the second equality, I applied the conditional Bayes theorem on the probability density function and used the fact that signals $s_{i,t}, \ldots, s_{i,t_0}$ as a group is conditionally independent with prices $p_0, \ldots, p_{t_0}$. In the third inequality, I further used the fact that the signals themselves are conditionally independent.

Substitute in the expressions of $\theta_{it}$ for $t = t_0 + \Delta t, \ldots, T - \Delta t$ and simplify,

$$
E^i_{t_0}[-\exp(-AW_{iT})]
= \int -\exp \left[-AW_{i,t_0} - \left( A\theta_{i,t_0} - \sum_{t=0}^{t_0-\Delta t} a_{iu}s_{i,u+\Delta t}\Delta t \right)(p_{0+t\Delta t} - p_{it}) + A \sum_{t=t_0}^{T-\Delta t} C(a_{it})\Delta t \right)
\cdot \prod_{t=0}^{t_0-\Delta t} \frac{1}{\sqrt{2\pi(a_{it}\Delta t)^{-1}}} \exp \left(-\frac{a_{it}\Delta t}{2}(s_{i,t+\Delta t} - p_{it})^2 \right) dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0})
$$

(A6)

Investors’ utility maximization problems are similar despite the differences in private
signals received. In the above integral, $s_{i,t_0}, \ldots, s_{i,t_0}$ only appears as coefficients of $p_{t_0} - p_t$. Taking the first order condition, $A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu} s_{i,u+\Delta t} \Delta t$ must be identical across investors. Define
\[ \theta_{i,t_0} = \frac{1}{A} \left( A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu} s_{i,u+\Delta t} \Delta t \right). \] (A7)

$\theta_{i,t_0}$ represents the common component of asset demand and only relies on the price history $p_0, \ldots, p_{t_0}$. This completes the backward induction for the asset demand equation.

Next, I prove that (13) is correct for date $t = t_0 - \Delta t$. Attention at this date is decided without the knowledge of date $t_0$ information. Substitute in (A6) and differentiate date $t_0 - \Delta t$ expected utility $E_{t_0-\Delta t} \left[ E_{t_0} [U(W_T)] \right]$ with respect to $a_{i,t_0-\Delta t}$:
\[ E \left[ U'(W_T) \left( A \cdot C'(a_{i,t_0-\Delta t}) \Delta t - \frac{\Delta t}{2} (y^2 - p_{i,t_0}^2) \right) \right] | p_0, \ldots, p_{t_0-\Delta t} = 0. \] (A8)

Apply the law of iterated expectations and use the fact that $p_{t_0} = E_{t_0}^*[y]$:
\[ C'(a_{i,t_0-\Delta t}) = \frac{1}{A} E_{t_0-\Delta t}^* \left[ E_{t_0}^* (y^2 - p_{i,t_0}^2) \right] = \frac{1}{A} E_{t_0-\Delta t}^*[v]. \] (A9)

This completes the backward induction for the attention equation.

**Proof of Lemma 2.**

I arrive at (21), (23), and (24) by taking first difference of $\theta_t, \chi_t$, and $\tau_t$ from equations (18)-(20).

Let $G_t(y)$ denote the posterior distribution of $y$ after observing all public information up to date $t$. $m_t$ and $h_t$ respectively represent the mean and variance of this distribution:
\[ m_t = E[y|p_0, p_1, \ldots, p_t] = \int y \, dG_t(y), \]
\[ h_t = \text{Var}[y|p_0, p_1, \ldots, p_t] = \int (y - m_t)^2 \, dG_t(y). \] (A10)

The signal $A(a_t \Delta t)^{-1}(\theta_t + \Delta t - \theta_t)$ communicates payoff $y$ perturbed by a normal noise with precision $A^{-2} \sigma_y^{-2} a_t^2 \Delta t$. It represents new public information arrived at date $t + \Delta t$.
and is independent of all previous public signals. Applying Bayes formula,

\[
m_{t+\Delta t} = E[y|p_0, p_1, \ldots, p_{t+\Delta t}] = \int \exp \left( -\frac{1}{2} A^{-2} \sigma_z^{-2} a^2 \Delta t \left( y - A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) \right)^2 \right) y dG_t(y)
\]

\[
= \int \exp \left( -\frac{1}{2} A^{-2} \sigma_z^{-2} a^2 \Delta t \left( y - A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) \right)^2 \right) dG_t(y)
\]

\[
= \left( 1 - A^{-2} \sigma_z^{-2} a^2 \Delta t \right) m_t + A^{-2} \sigma_z^{-2} a^2 \Delta t \cdot A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) + o(\Delta t).
\]

Substituting in the expression of \( A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) \) from (16), I obtain (22).

**Proof of Lemma 3.**

First I prove that \( \xi_{i,t,T} \) defined in (29) is a valid stochastic discount factor.

At date \( t \), investor \( i \) makes portfolio choice \( \theta_{it} \) to maximize her expected utility \( E_t[U(W_{iT})] \), where

\[
W_{iT} = W_{it} + \sum_{u=t}^{T-\Delta t} \theta_{iu}(p_{u+\Delta t} - p_u) + A \sum_{u=t}^{T-\Delta t} C(a_{iu}) \Delta t.
\]

(A12)

The first-order condition with respect to \( \theta_{iu} \) suggests that

\[
E_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = 0.
\]

(A13)

Apply the law of iterated expectations:

\[
E_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = E_t^i \left[ E_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] \right] = 0,
\]

(A14)

\[
E_t^i[U'(W_{iT})(y - p_t)] = \sum_{u=t}^{T-\Delta t} E_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = 0,
\]

(A15)

Substitute in the definition of \( \xi_{i,t,T} \),

\[
E_t^i[\xi_{i,t,T}(y - p_t)] = 0.
\]

(A16)

\( E_t^i \) represents the conditional expectation using investor \( i \)’s private information set at time \( t \). \( E_t \) represents the conditional expectation using the public information set (the price history). The private information set contains the public one. Apply law of iterated expectations:

\[
E_t[\xi_{i,t,T}(y - p_t)] = E_t[E_t^i[\xi_{i,t,T}(y - p_t)] = 0.
\]

(A17)
Now consider \( \xi_{t,T} = \mathbb{E} \left[ \xi_{t,T}^i | p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \right] \).

\[
\begin{align*}
E_t[\xi_{t,T}(y - p_t)] &= E_t \left[ E \left[ \xi_{t,T}^i | p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \right] (y - p_t) \right] \\
&= E_t \left[ E \left[ \xi_{t,T}^i (y - p_t) | p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \right] \right] \\
&= E_t[\xi_{t,T}^i(y - p_t)] = 0. 
\end{align*}
\]

(A18)

In the second equality, I used the fact that both \( p_t \) and \( y \) are measurable with respect to \( p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \). In the third equality I applied the law of iterated expectations. As a result, \( \xi_{t,T} \) prices the risky asset correctly. Furthermore, \( E_t[\xi_{t,T}^i] = 1 \). This completes the proof that \( \xi_{t,T} \) is a valid SDF.

Next, I prove that \( \xi_{t,T} \) has the expression given in (30). From the definition of \( \xi_{t,T}^i \),

\[
\xi_{t,T}^i = \frac{U'(W_{iT})}{E_t[U'(W_{iT})]} = -\frac{A U(W_{iT})}{E_t[-AU(W_{iT})]} = \frac{U(W_{iT})}{E_t[U(W_{iT})]}.
\]

(A19)

\( U(W_{iT}) \) is given by:

\[
U(W_{iT}) = -\exp \left[ -\sum_{t=0}^{T-\Delta t} \left( A \theta_t + \sum_{u=0}^{u-\Delta t} a_i s_{i,u+\Delta t} (p_{u+\Delta t} - p_u) \right) + A \sum_{t=0}^{T-\Delta t} C(a_{iu}) \Delta t \right].
\]

(A20)

The derivation of \( E_t^i[U(W_{iT})] \) is similar to (A6):

\[
E_t^i[U(W_{iT})] = \int \! \exp \left[ -\sum_{t=0}^{T-\Delta t} \left( A \theta_t + \sum_{u=0}^{u-\Delta t} a_i s_{i,u+\Delta t} (p_{u+\Delta t} - p_u) \right) + A \sum_{t=0}^{T-\Delta t} C(a_{iu}) \Delta t \right] \\
\cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu} \Delta t)^{-1}}} \exp \left( -\frac{a_{iu} \Delta t}{2} (s_{i,u+\Delta t} - p_t)^2 \right) \cdot \frac{dG(y, p', s')|p_0, \ldots, p_t}{\pi(s_{i,t}, \ldots, s_{it}|p_0, \ldots, p_t)}
\]

\[
= \int \! \exp \left[ -\sum_{t=0}^{T-\Delta t} \left( A \theta_t (p_{u+\Delta t} - p_u) + \frac{1}{2} \tau_u (p_{u+\Delta t} - p_u)^2 - A \cdot C(a_{iu}) \Delta t \right) \right] \\
\cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu} \Delta t)^{-1}}} \exp \left( -\frac{a_{iu} \Delta t}{2} (s_{i,u+\Delta t} - p_t)^2 \right) \cdot \frac{dG(y, p', s')|p_0, \ldots, p_t}{\pi(s_{i,t}, \ldots, s_{it}|p_0, \ldots, p_t)}
\]

\[
= -\exp(f_t) \cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu} \Delta t)^{-1}}} \exp \left( -\frac{a_{iu} \Delta t}{2} (s_{i,u+\Delta t} - p_t)^2 \right) \cdot \frac{1}{\pi(s_{i,t}, \ldots, s_{it}|p_0, \ldots, p_t)}.
\]

(A21)

In the second equality, I applied the conditional Bayes theorem and then integrated over the signals \( s_{i,t+\Delta t}, \ldots, s_{i,T-\Delta t} \), similar to (A5) and (A6). In the third equality, I used
the definition of \( f_t \) (31). Therefore,

\[
\xi_{t,T} = \mathbb{E} \left[ \frac{U(W_{tT})}{\mathbb{E}[U(W_{tT})]} | p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \right]
\]

\[
= \mathbb{E} \left[ \exp \left[ - \sum_{u=t}^{T-\Delta t} \left( A\theta_t + \sum_{u_0=0}^{u-\Delta t} a_{i,u_0} s_{\Delta t,u_0} (p_{u+\Delta t} - p_u) \right) + A \sum_{u=t}^{T-\Delta t} C(a_{iu}) \Delta t - f_t \right] \cdot \pi(s_{\Delta t}, \ldots, s_t | p_0, \ldots, p_t) \cdot \prod_{u=0}^{t-\Delta t} \left( \frac{1}{\sqrt{2\pi(a_{iu})^{-1}}} \exp(-\frac{a_{iu} \Delta t}{2}(s_{i,u+\Delta t} - p_t)^2) \right)^{-1} \right] \Big| p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y \right].
\]

Applying conditional Bayes theorem and integrating over the signals once again, I arrive at (30).

**Proof of Proposition 2.**

First, I prove the expression of \( \mu_{pt} \) and \( \sigma_{pt} \) given in (36). Apply Taylor series expansion to \( p(t + \Delta t, \theta_t, m_t, \chi_t, \tau_t) \) around \((t, \theta_t, m_t, \chi_t, \tau_t)\),

\[
p(t + \Delta t, \theta_t, m_t, \chi_t, \tau_t) = p(t, \theta_t, m_t, \chi_t, \tau_t) + \frac{\partial p}{\partial t}(\Delta t) + \frac{\partial p}{\partial \theta_t}(\theta_t - \theta_t) + \frac{\partial p}{\partial m_t}(m_t + \Delta t - m_t) + \frac{\partial p}{\partial \chi_t}(\chi_t + \Delta t - \chi_t)
\]

\[
+ \frac{\partial^2 p}{\partial \theta_t^2}(\theta_t + \Delta t - \theta_t) + \frac{1}{2} \frac{\partial^2 p}{\partial \theta_t^2}(\theta_t + \Delta t - \theta_t) + \frac{1}{2} \frac{\partial^2 p}{\partial \theta_t^2}(m_t + \Delta t - m_t)
\]

\[
+ \frac{\partial^2 p}{\partial \theta_t \partial m_t}(m_t + \Delta t - m_t) + o(\Delta t).
\]

Substituting in (21)-(24) and use the fact the expectation of payoff \( E_t[y] \) is \( m_t \), the conditional expectations of state variable increments are given by:

\[
E_t[\theta_{t+\Delta t} - \theta_t] = E_t[-A^{-1}a_{1} y \Delta t + z_{t+\Delta t} - z_t] = -A^{-1}a_{1} m_{t} \Delta t,
\]

\[
E_t[m_{t+\Delta t} - m_t] = E_t[A^{-2} \sigma_{1}^2 h_t(y - m_t) \Delta t - A^{-1} \sigma_{1}^2 a_{1} h_t(z_{t+\Delta t} - z_t) + o(\Delta t)]
\]

\[
= A^{-2} \sigma_{1}^2 a_{1}^2 \Delta t E_t[y - m_t] + o(\Delta t) = o(\Delta t),
\]

\[
E_t[\chi_{t+\Delta t} - \chi_t] = A^{-2} \sigma_{2}^2 \Delta t,
\]

\[
E_t[\tau_{t+\Delta t} - \tau_t] = a_t \Delta t.
\]

The conditional variance of \( \theta_{t+\Delta t} - \theta_t \) is:

\[
\text{Var}_t[\theta_{t+\Delta t} - \theta_t] = \text{Var}_t[-A^{-1} a_{1} y \Delta t + z_{t+\Delta t} - z_t]
\]

\[
= \text{Var}_t[-A^{-1} a_{1} y \Delta t] + \text{Var}_t[z_{t+\Delta t} - z_t]
\]

\[
= A^{-2} a_{1}^2 \text{Var}_t[y] \cdot (\Delta t)^2 + \sigma_{1}^2 \Delta t = \sigma_{1}^2 \Delta t + o(\Delta t).
\]

(25)
Similarly,
\[
\Var_t[m_{t+\Delta t} - m_t] = A^{-2}\sigma^{-4} a_t^2 h_t^2 \Var_t[z_{t+\Delta t} - z_t] = A^{-2}\sigma^{-2} a_t^2 h_t^2 \Delta t,
\]
\[
\Cov_t[\theta_{t+\Delta t} - \theta_t, m_{t+\Delta t} - m_t] = A^{-1}\sigma^{-2} a_t h_t \Var_t[z_{t+\Delta t} - z_t] = A^{-1} a_t h_t \Delta t.
\] (A27)

Substituting (A24) and (A25)-(A27) into (34) and simplify, I arrive at the expression of drift $\mu_{pt}$ and volatility $\sigma_{pt}$.

Next, I derive the system of partial differential equations (37)-(39). Let $\xi_{t,t+\Delta t}$ denote the stochastic discount factor from date $t$ to $t+\Delta t$:
\[
\xi_{t,t+\Delta t} = \frac{\xi_{t,T}}{\xi_{t,t+\Delta t, T}} = \exp\left(-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t - f_t + f_{t+\Delta t}\right).
\] (A28)

Because the interest rate is zero, $E_t[\xi_{t,t+\Delta t}]$ is equal to 1. From (26),
\[

p_t = E_t[\xi_{t,t+\Delta t} p_{t+\Delta t}] = E_t[\xi_{t,t+\Delta t}] E_t[p_{t+\Delta t}] + \Cov_t[\xi_{t,t+\Delta t}, p_{t+\Delta t}]

= p_t + \mu_{pt} \Delta t + \Cov_t\left[-A\theta_t p_{t+\Delta t} - \frac{1}{2}\tau_t p_{t+\Delta t}^2 + f_{t+\Delta t}, p_{t+\Delta t}\right] + o(\Delta t)

= p_t + \left(\mu_{pt} + \sigma_{pt} \sigma_{ft} - (A\theta_t + \tau_t p_t) \sigma_{pt}^2\right) \Delta t + o(\Delta t).
\] (A29)

The first equation (37) in the system of PDEs is obtained by taking the limit $\Delta t \to 0$. Similarly, the second equation (38) comes from
\[
v_t = E_t[\xi_{t,t+\Delta t} v_{t+\Delta t}] + \Var_t[p_{t+\Delta t}]

= E_t[\xi_{t,t+\Delta t} v_{t+\Delta t}] + \Var_t[p_{t+\Delta t}] + o(\Delta t)

= v_t + \left(\mu_{vt} + \sigma_{vt} \sigma_{ft} - (A\theta_t + \tau_t p_t) \sigma_{vt}^2 + \sigma_{pt}^2\right) \Delta t + o(\Delta t).
\] (A30)

Apply Taylor expansion to $\xi_{t,t+\Delta t} = \exp(\ln(\xi_{t,t+\Delta t}))$ around $\ln(\xi_{t,t+\Delta t}) = 0$:
\[
\xi_{t,t+\Delta t} = 1 + \ln(\xi_{t,t+\Delta t}) + \frac{1}{2}(\ln(\xi_{t,t+\Delta t}))^2 + o(\Delta t)
\]
\[
E_t[\xi_{t,t+\Delta t}] = 1 + E_t\left[-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t - f_t + f_{t+\Delta t}\right]

+ \frac{1}{2}E_t\left[\left(-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A \cdot C(a_t) \Delta t - f_t + f_{t+\Delta t}\right)^2\right] + o(\Delta t)

= 1 + \left[-(A\theta_t + \tau_t p_t) \mu_{pt} + A \cdot C(a_t) + \mu_{ft} + \frac{1}{2}(-(A\theta_t + \tau_t p_t) \sigma_{pt} + \sigma_{ft})^2\right] \Delta t + o(\Delta t).
\] (A31)

Taking the limit $\Delta t \to 0$ and substitute in (37) to eliminate $\mu_{pt}$, I arrive at (39). The equation that relates attention to risk-neutral variance (13) converges to equation (40) in the continuous-time limit.
The terminal conditions for this system of equations come from the continuous-time limit of \( p_{T-\Delta t} \), \( v_{T-\Delta t} \), and \( f_{T-\Delta t} \):

\[
\begin{align*}
p_{T-\Delta t} &= E_{T-\Delta t} \left[ \exp \left( -A\theta_{T-\Delta t}(y - p_{T-\Delta t}) - \frac{1}{2} \tau_{T-\Delta t}(y^2 - p_{T-\Delta t}^2) \right) \Delta t - f_{T-\Delta t} \right] \cdot y \\
v_{T-\Delta t} &= E_{T-\Delta t} \left[ \exp \left( -A\theta_{T-\Delta t}(y - p_{T-\Delta t}) - \frac{1}{2} \tau_{T-\Delta t}(y^2 - p_{T-\Delta t}^2) \right) \Delta t - f_{T-\Delta t} \right] \cdot (y - p_{T-\Delta t})^2 \\
f_{T-\Delta t} &= \log \left( E_{T-\Delta t} \left[ \exp \left( -A\theta_{T-\Delta t}(y - p_{T-\Delta t}) - \frac{1}{2} \tau_{T-\Delta t}(y^2 - p_{T-\Delta t}^2) \right) \right] \right). \quad (A32)
\end{align*}
\]

**Proof of Proposition 3.**

Risk-neutral variance at date \( t \) and \( t + \Delta t \) is related by equation (27):

\[
E_{t}^{*}[v_{t+\Delta t} - v_{t}] = -\text{Var}_{t}^{*}[p_{t+\Delta t}] = - \left( E_{t}[\xi_{t,t+\Delta t} p_{t+\Delta t}^2] - p_{t}^2 \right) \\
= - (E_{t}[p_{t+\Delta t}^2] - p_{t}^2) - \text{Cov}_{t}[\xi_{t,t+\Delta t}, p_{t+\Delta t}^2] \\
= - \sigma_{pt}^2 \Delta t + o(\Delta t), \quad (A33)
\]

As a result, the instantaneous drift of \( v_{t} \) in the risk-neutral measure is given by:

\[
\mu_{vt}^* = \lim_{\Delta t \to 0} \frac{E_{t}^{*}[v_{t+\Delta t} - v_{t}]}{\Delta t} = - \sigma_{pt}^2 \\
= - \left( \frac{\partial p}{\partial m} A^{-1} \sigma_{z}^{-1} a_{t} h_{t} - \frac{\partial p}{\partial \theta} \sigma_{z} \right)^2. \quad (A34)
\]

Since price is increasing in \( m_{t} \) and decreasing in \( \theta_{t} \), \( \mu_{vt}^* \) is decreasing in attention:

\[
\frac{\partial \mu_{vt}^*}{\partial a_{t}} = 2 \frac{\partial p}{\partial m} \frac{\partial p}{\partial \theta} A^{-1} h_{t} - 2 \left( \frac{\partial p}{\partial m} \right)^2 A^{-2} \sigma_{z}^{2} a_{t} h_{t}^2 < 0. \quad (A35)
\]

I introduce Lemma A1 and Lemma A2 to prove Proposition 4.

**Lemma A1.** \( \partial f_{t} / \partial \theta_{t} \) is given by:

\[
\frac{\partial f_{t}}{\partial \theta_{t}} = (A\theta_{t} + \tau_{t} p_{t}) \frac{\partial p_{t}}{\partial \theta_{t}} + E_{t}^{*} \left[ \int_{t}^{T} \frac{\partial^2 p_{u}}{\partial \theta_{u}^{2}} A \sigma_{z}^{2} du \right], \quad (A36)
\]

**Proof Lemma A1.**

I start from the definition of \( f_{t} \) in discrete time (31):

\[
f_{t} = \ln E_{t} \left[ \exp \left( - \sum_{u=t}^{T-\Delta t} A\theta_{u} (p_{u+\Delta t} - p_{u}) + \frac{1}{2} \tau_{u} (p_{u+\Delta t}^2 - p_{u}^2) - A \cdot C(a_{u}) \Delta t \right) \right]. \quad (A37)
\]
Differentiate $f_t$ in (31) with respect to $p_t$,
\[ \frac{\partial f_t}{\partial p_t} = (A\theta_t + \tau_t p_t). \]

An increase in $\theta_t$ implies the same change in $\theta_u$ for $u = t + \Delta t, \ldots, T - \Delta t$. From the dynamics of state variables (21),
\[ \frac{\partial \theta_u}{\partial \theta_t} = 1, \quad u = t + \Delta t, \ldots, T - \Delta t. \] (A38)

Now differentiate $f_t$ in (31) with respect to $\theta_u$
\[ \frac{\partial f_t}{\partial \theta_u} = \frac{1}{f_t} \mathbb{E}_t \left[ \exp \left( - \sum_{u=t}^{T-\Delta t} \left[ A\theta_u (p_u + \Delta t - p_u) + \frac{1}{2} \tau_u (p_u^2 - p_u) - A \cdot C(a_u) \Delta t \right] \right) \left( -A(p_u + \Delta t - p_t) + (A\theta_u - A\theta_{u-\Delta t} + a_{u-\Delta t} \Delta t p_u) \frac{\partial p_u}{\partial \theta_u} \right) \right]. \]
\[ = \mathbb{E}_t \left[ -A(p_u + \Delta t - p_t) + (A\theta_u - A\theta_{u-\Delta t} + a_{u-\Delta t} \Delta t p_u) \frac{\partial p_u}{\partial \theta_u} \right] + \text{Cov}_t^* \left[ (A\theta_u - A\theta_{u-\Delta t} + a_{u-\Delta t} \Delta t p_u), \frac{\partial^2 p_u}{\partial \theta_u^2} (\theta_u - \theta_{u-\Delta t}) \right] + o(\Delta t) \]
\[ = \frac{\partial^2 p_u}{\partial \theta_u^2} A \sigma_z^2 \Delta t + o(\Delta t). \] (A39)

The third equality in the above expression follows from the fact that price is a martingale under the risk-neutral measure and that $\mathbb{E}_t^*[A\theta_u - A\theta_{u-\Delta t} + a_{u-\Delta t} \Delta t p_u] = 0$. From (A38) and (A39), I obtain (A36).

**Lemma A2.** $\partial f_t/\partial m_t$ is given by:
\[ \frac{\partial f_t}{\partial m_t} = (p_t - m_t) h_t^{-1} + (A\theta_t + \tau_t p_t) \frac{\partial p_t}{\partial m_t} - \mathbb{E}_t^* \left[ \int_t^T \frac{X_u}{\chi_t} \frac{\partial^2 p_u}{\partial m_u^2} a_u h_u \right]. \] (A40)

**Proof Lemma A2.**

From equation (16), $A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$ is a public signal of $y$ with precision $A^{-2} \sigma_z^{-2} a_t^2 \Delta t$. $\chi_t$ represents the aggregate precision of all public signals. Let $\eta_t$ denote the average of
public signals using precisions $A^{-2}\sigma_z^{-2}a_i^2\Delta t$ as weights:

$$\chi_t = \sum_{u=0}^{t-\Delta t} A^{-2}\sigma_z^{-2}a_u^2\Delta t,$$  \hfill (A41)

$$\eta_t = \sum_{u=0}^{t-\Delta t} A^{-2}\sigma_z^{-2}a_u^2\Delta t \cdot \chi_{t-\Delta t}. \hfill (A42)$$

Because public signals in different periods are independent, they are informationally equivalent to a signal $\eta_t$ with precision $\chi_t$. Applying Bayes formula, the expected payoff $m_t$ could be expressed as:

$$m_t = \frac{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot y \cdot dG(y) \right]}{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot dG(y) \right]},$$

\hfill (A43)

Differentiate $m_t$ with respect to $\eta_t$:

$$\frac{dm_t}{d\eta_t} = \frac{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot y \cdot \chi_t(y - \eta_t) \cdot dG(y) \right]}{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot dG(y) \right]} - m_t \cdot \frac{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot \chi_t(y - \eta_t) \cdot dG(y) \right]}{\int \left[ \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot dG(y) \right]}$$

$$= E_t[y \cdot \chi_t(y - \eta_t)] - m_t E_t[\chi_t(y - \eta_t)]$$

$$= \chi_t \left[ E_t[y^2] - m_t E_t[y] \right] = \chi_t h_t.$$ \hfill (A44)

Similarly, $f_t$ could be expressed as:

$$f_t = \ln \left( \int \exp \left( -\sum_{u=1}^{T-\Delta t} \left[ A\theta_u(p_{u+\Delta t} - p_u) + \frac{1}{2}\tau_u(p_{u+\Delta t}^2 - p_u^2) - A \cdot C(a_u)\Delta t \right] - \frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot dG(y) \right)$$

$$- \ln \left( \int \exp \left( -\frac{\chi_t}{2} (y - \eta_t)^2 \right) \cdot dG(y) \right) \hfill (A45)$$

Differentiate $f_t$ with respect to $\eta_t$:

$$\frac{df_t}{d\eta_t} = \chi_t E_t[y - \eta_t] - \chi_t E_t[y - \eta_t] = \chi_t (p_t - m_t).$$ \hfill (A46)

(A44) and (A46) give rise to the first term in (A40). The second term is similar to Lemma 1.
An increase in $m_t$ implies less than one-to-one change in $m_u$ for $u = t + \Delta t, \ldots, T - \Delta t$. From the dynamics of state variables (22),

$$\frac{dm_u}{dm_t} = \frac{\chi_u}{\chi_t} + o(\Delta t), \quad u = t + \Delta t, \ldots, T - \Delta t. \tag{A47}$$

Similar to (A39),

$$\frac{\partial f_t}{\partial m_u} = -\frac{\partial^2 p_u}{\partial m^2_u} a_u h_u + o(\Delta t). \tag{A48}$$

The above expression gives the third term in (A40).

**Proof of Proposition 4.**

Using (37) and (36), the instantaneous expected return $\mu_{pt}$ is given by:

$$\mu_{pt} = \sigma_{pt}( - \sigma_{ft} + (A\theta_t + \tau_t p_t)\sigma_{pt})$$

$$= \left(\frac{\partial p_t}{\partial m_t} A^{-1} \sigma_z^{-1} h_t a_t - \frac{\partial p_t}{\partial \theta_t} \sigma_z\right) (-\sigma_{ft} + (A\theta_t + \tau_t p_t)\sigma_{pt})$$

$$= \left(\frac{\partial p_t}{\partial m_t} A^{-1} \sigma_z^{-1} h_t a_t - \frac{\partial p_t}{\partial \theta_t} \sigma_z\right) \left(\frac{\partial f_t}{\partial \theta_t} \sigma_z - \frac{\partial^2 f_t}{\partial m_t^2} A^{-1} \sigma_z^{-1} h_t a_t + (A\theta_t + \tau_t p_t)\sigma_{pt}\right). \tag{A49}$$

Combine (A49) with the results from Lemma A1 and Lemma A2, I arrive at (48).

**Proof of Proposition 5.**

Define $\eta_{T-\Delta t}$ similar to $\eta_t$ in (A42). Public signals from date 0 to date $T - \Delta t$ are informationally equivalent to a signal $\eta_{T-\Delta t}$ with precision $\chi_{T-\Delta t}$. Similar to (A44),

$$\frac{dm_{T-\Delta t}}{d\eta_{T-\Delta t}} = \chi_{T-\Delta t} \left[ E_{T-\Delta t}[y^2] - m_{T-\Delta t} E_{T-\Delta t}[y] \right] = \chi_{T-\Delta t} h_{T-\Delta t}. \tag{A50}$$

Applying the Bayes formula, the risk-neutral variance $v_{T-\Delta t}$ can be expressed as follows:

$$v_{T-\Delta t} = \int \left[ \xi_{T-\Delta t,T} \cdot \exp \left( -\frac{\chi_{T-\Delta t}}{2} (y - \eta_{T-\Delta t})^2 \right) \cdot (y - p_{T-\Delta t}^3) dG(y) \right]$$

$$\int \left[ \xi_{T-\Delta t,T} \cdot \exp \left( -\frac{\chi_{T-\Delta t}}{2} (y - \eta_{T-\Delta t})^2 \right) dG(y) \right]. \tag{A51}$$

And the partial derivative of $v_{T-\Delta t}$ to $\eta_{T-\Delta t}$ is:

$$\frac{\partial v_{T-\Delta t}}{\partial \eta_{T-\Delta t}} = \chi_{T-\Delta t} E_{T-\Delta t}^* [ (y - p_{T-\Delta t}^3)].$$

Combine (A50) and (A52), I arrive at (50).