Dynamic Principal Agent Models: A Continuous Time Approach

Lecture III

Dynamic Financial Contracting II - Convergence to Continuous Time
(Biais et al. 2007)

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Motivation

- So far we have discussed models that are formulated directly in continuous time and studied how to solve these using martingale techniques.
- Still, several other models are formulated in discrete time e.g. Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007).
- Natural question: Convergence of discrete time models to continuous time limit:
  - Build intuition (make precise timing and info structure within each period, "extensive form"),
  - Show some advantages of cont. time modelling.

Outline

1. The discrete time model:
   - Setup,
   - Static model,
   - Optimal Contract with equally patient and impatient agent.

2. The continuous time limit:
   - Convergence Results,
   - Discussion and robustness.

Note: We will not provide detailed proofs of the results in this part of the lecture. The discussion will be on an intuitive level. Rigorous proofs of the presented results can be found in Biais et al. (2004, 2007).

Part 1:
The Discrete Time Model.

Model Setup

- Time is discrete with periods of length $h > 0$ indexed by $n \in \mathbb{N}$, so real time is $t = nh$.
- Risk-neutral principal with discount rate r.
- Risk-neutral agent with discount rate $\rho \geq r$.
- Project can be managed by agent only and requires initial investment of $K \geq 0$.
- Agent is protected by limited liability and has limited wealth $B < K$.

Model Setup

- Project yields a stream of i.i.d. cash flows y_{nh}^h:

$$y_{nh}^h = \begin{cases} y_+^h = \mu h + \sigma\varepsilon_+\sqrt{h} & \text{with prob. } p \\ y_-^h = \mu h + \sigma\varepsilon_-\sqrt{h} & \text{with prob. } 1-p \end{cases},$$

with $\varepsilon_+ = \sqrt{\frac{1-p}{p}}$ and $\varepsilon_- = -\sqrt{\frac{p}{1-p}}$, so

$$E\left[y_{nh}^h\right] = \mu h,$$
$$Var\left[y_{nh}^h\right] = \sigma^2 h.$$

- Assume h small enough such that $y_-^h < 0$ ("operating expenses").
- Liquidation value and agent's outside option equal zero.

Model Setup

- Principal does not observe y^h_{nh} directly, but only the agent's report $\widehat{y}^h_{nh}$.
  $\rightarrow$ Cash flow diversion problem.
- Agent profits at rate $\lambda \in (0, 1]$ from any unit diverted.
- Agency problem is "severe" for h close enough to zero:

$$\underbrace{p\lambda\left(y^h_+ - y^h_-\right)}_{expected\ benefit\ from\ diversion} > \underbrace{\mu h}_{expected\ cash\ flow} .$$

  $\rightarrow$ Project can not be financed in the static model ("credit rationing") for $B = 0$.
- Note: The formal analysis of this cash flow diversion model is identical to a hidden effort model with binary effort choice and private benefits from shirking, if it is optimal to request effort in all contingencies.

Detour: The Static Model

- Let us have a look at the one-shot version of this model.
- Agent has to borrow K from investors in order to run the project ($B = 0$ for simplicity).
- The project generates cash flow of $y_+ > 0$ with probability p, while with prob $(1-p)$ cash flow is $y_- < y_+$.
- Running the project is efficient, i.e., it has a positive NPV:

$$\mu h = p y_+ + (1-p)\, y_- > K.$$

Detour: The Static Model

- Focus on truthtelling contracts (revelation principle): Agent delivers true cash flow to investors and receives a contingent transfer c_+, $c_- \geq 0$.
- If the principal did not request truthtelling, his expected profit would be

$$\underbrace{\mu h - K}_{\textit{net cash flow}} - \underbrace{p\lambda\left(y_+ - y_-\right)}_{\textit{utility from diversion}} - \underbrace{p\left(1-\lambda\right)\left(y_+ - y_-\right)}_{\textit{cost of diversion}},$$

  which is smaller than what can be achieved under truthtelling.
- The agent tells the truth if

$$c_+ \geq \lambda(y_+ - y_-) + c_-.$$

  - Optimal to set $c_- = 0$, so, $p\lambda\left(y_+ - y_-\right)$ is minimum rent required to induce truthtelling.

Detour: The Static Model

- Accordingly, the maximum (expected) income for investors is

$$\mu h - p\lambda \left(y_+ - y_- \right).$$

- This is consistent with the investors' participation constraint if and only if

$$\mu h - p\lambda \left(y_+ - y_- \right) \geq K.$$

$\longrightarrow$ Credit rationing in the static model if the agency problem is "severe".

- Financing problem is relaxed if:
  - Agent has initial wealth $B > 0$,
  - Randomization over setting up the firm is allowed.

Detour: The Static Model

- With $B > 0$, agent needs to raise only $K - B$:
  - Still optimal to set $c_- = 0$ and the agent's incentive and participation constraints imply

$$c_+ \geq \max\left\{\lambda(y_+ - y_-), \frac{B}{p}\right\}.$$

  - So, the principal participates if

$$\mu h - p \max\left\{\lambda(y_+ - y_-), \frac{B}{p}\right\} \geq K - B.$$

- Agent's "stake" in the firm relaxes the financing problem.

Detour: The Static Model

- Randomization further relaxes the problem:
  - Consider a small value of B for which financing with prob 1 is not feasible (the incentive constraint binds).
- Starting the firm with prob x is feasible if

$$x\left(\mu h - p\lambda(y_+ - y_-)\right) \geq xK - B.$$

$\longrightarrow$ Maximal initial "scale" of project:

$$x = \min\left\{\frac{B}{K - (\mu h - p\lambda(y_+ - y_-))}, 1\right\}.$$

- Takeaway:
  - Credit rationing in static model if agency problem is severe.
  - Agent's wealth and the possibility for randomization relax the financing problem.
  - Next: Repeated interaction may allow financing even if agency problem is severe and agent has no initial wealth.

The Dynamic Model - Timing

- Back to the dynamic model (stationary, infinite horizon case).
- At any date nh, given the history of reports $\left\{\widehat{y}^h_{mh}\right\}_{m=0}^{n-1}$:
  1. The project is continued with probability x^h_{nh},
  2. The principal pays operating costs $-\underline{y}^h$,
  3. Given y^h_{nh}, the agent reports $\widehat{y}^h_{nh}$ and makes payment $\widehat{y}^h_{nh} - \underline{y}^h$ to the principal,
  4. Based on $\left\{\widehat{y}^h_{mh}\right\}_{m=0}^{n}$ the principal makes payment $c^h_{nh} \geq 0$ to the agent.

The Recursive Formulation

- The optimal long-term contract will be derived using dynamic programming with agent's expected discounted utility as only state variable (stationary case).
- Given continuation value w, the optimal contract specifies:
  1. Continuation probability $x \in [0, 1]$,
  2. Contingent transfers $(c_+, c_-) \in \mathbb{R}^2_+$,
  3. Contingent continuation values $(w_+, w_-) \in \mathbb{R}^2_+$.
     $\rightarrow$ Limited liability:

$$(LL) \qquad c_{+,-} \geq 0, \; w_{+,-} \geq 0.$$

- From the revelation principle it is w.l.o.g. to require truthful reporting.

The Principal's Problem

- Denote by $F^h(w)$ the principal's value function solving

$$F^h(w) = \max_{\substack{x \\ c_{+,-} \\ w_{+,-}}} \left\{ x \left[\mu h - pc_+ - (1-p)\,c_- + \frac{pF^h(w_+) + (1-p)\,F^h(w_-)}{1+rh} \right] \right\},$$

subject to (LL), consistency ("promise keeping")

$$(PK) \qquad w = x \left[pc_+ + (1-p)\,c_- + \frac{pw_+ + (1-p)\,w_-}{1+\rho h} \right],$$

and incentive compatibility

$$(IC) \qquad c_+ + \frac{w_+}{1+\rho h} \geq c_- + \frac{w_-}{1+\rho h} + \lambda \left(y_+^h - y_-^h \right).$$

- Compare with static IC: Possibility of deferred payment (the promise of future rents) relaxes incentive constraint.

An Alternative Representation

- Denote the social surplus for all $w \geq 0$ by $V^h(w) = w + F^h(w)$, which is independent of current transfers.
- The principal's problem can then be rewritten as

$$V^h(w) = \max_{x, w_+, w_-} \left\{ x \left[\begin{array}{c} \mu h + \frac{pV^h(w_+)+(1-p)V^h(w_-)}{1+rh} \\ -\frac{(\rho-r)h[pw_+ + (1-p)w_-]}{(1+rh)(1+\rho h)} \end{array} \right] \right\},$$

subject to

$$\begin{aligned} w &\geq x\left[\frac{w_-}{1+\rho h} + p\lambda\left(y_+^h - y_-^h\right)\right], \\ w &\geq x\left[\frac{pw_+ + (1-p)\,w_-}{1+\rho h}\right]. \end{aligned}$$

- It requires some work to eliminate $c_{+,-}$ from the constraints (without adding much intuition), see Biais (2007) Lemma 1 for a proof.

Solution to the Bellman Equation

Proposition 1: *There exists a unique continuous and bounded solution $V^h(w)$ to this programming problem, which is*

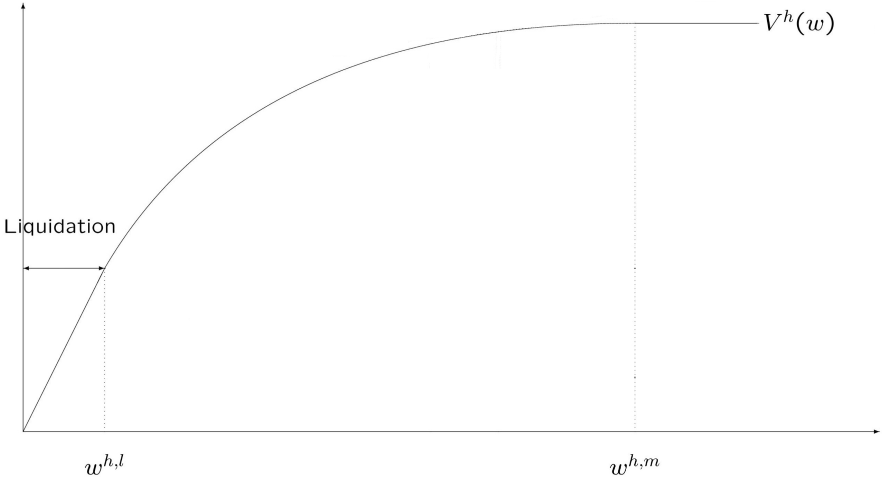
(i) *non-decreasing, concave and vanishes at zero,*

(ii) *linear over the region $w \in \left[0, w^{h,l}\right)$, where the project is continued with probability $w/w^{h,l}$,*

(iii) *strictly increasing over $\left[w^{h,l}, w^{h,m}\right)$, with continuation probability of 1,*

(iv) *constant over $\left[w^{h,m}, \infty\right)$, with continuation probability of 1.*

The Social Surplus



Intuition for Proposition 1

- Immediate from limited liability and

$$w \geq x \left[\frac{w_-}{1+\rho h} + p\lambda \left(y_+^h - y_-^h \right) \right],$$

that for $w < p\lambda \left(y_+^h - y_-^h \right)$ liquidation must occur with positive probability. As liquidation is inefficient, it arises only for low values of the agent's continuation value, $w < w^{h,l}$.

- The solution vanishes at zero because at $w = 0$ it is impossible to incentivize the agent for truthtelling ("too poor to be punished").
- The solution is non-decreasing, as a higher w reduces the risk of liquidation (benefit of deferred compensation).
- The solution is concave as the risk of inefficient liquidation endogenously creates an aversion to variations in w.

Intuition for Proposition 1

- If agent's stake in the company is large enough ($w \geq w^{h,m}$) there is no need to further defer compensation. Both constraints are slack and $w_{+,-}$ are chosen to solve

$$\max \left\{ \mu h + \frac{pV^h(w_+) + (1-p)\, V^h(w_-)}{1+rh} - \frac{(\rho - r)\, h\, [pw_+ + (1-p)\, w_-]}{(1+rh)\,(1+\rho h)} \right\}.$$

- Optimality implies that $w_+ = w_- =: w^{h,r}$, which is defined by

$$w^{h,r} \in \arg\max_{w} \left\{ V^h(w) - \frac{(\rho - r)\, hw}{(1+\rho h)} \right\},$$

$\rightarrow V^h(w)$ is constant for $w \geq w^{h,m}$.

- Optimal contract for large w crucially depends on relative impatience:
  - If $\rho = r$, boundary $w^{h,r} = w^{h,m}$ is absorbing,
  - If $\rho > r$, reflecting boundary $w^{h,r} < w^{h,m}$.

The Optimal Contract with Equally Patient Agent

- Intuitively, when $\rho = r$ there are no costs in delaying compensation and recapitalizing the promised rewards at rate r.

  $\rightarrow$ *Never optimal to make direct payments before enough profits have been accumulated to finance the incentive costs without ever relying on the liquidation threat.*

- The present value of these incentive costs is given by

$$w^{h,m}_{\rho=r} = \sum_{n=0}^{\infty} \left(\frac{1}{1+rh}\right)^n p\lambda \left(y^h_+ - y^h_-\right) = \frac{1+rh}{rh} p\lambda \underbrace{\left(y^h_+ - y^h_-\right)}_{=\sigma\sqrt{h}/\sqrt{p(1-p)}} .$$

- If $w^{h,m}_{\rho=r}$ is reached, firm is operated with certainty forever (first-best).

- As $\lim_{h\to 0} w^{h,m}_{\rho=r} = \infty$, the case with $\rho = r$ is not viable for continuous time analysis.

The Optimal Contract with Equally Patient Agent

Proposition 2: *Suppose $\rho = r$, then $w^{h,l}_{\rho=r}$ and $w^{h,m}_{\rho=r}$* are given by*:*

$$
\begin{aligned}
w^{h,l}_{\rho=r} &= p\lambda(y^h_+ - y^h_-), \\
w^{h,m}_{\rho=r} &= \frac{1+rh}{rh} p\lambda(y^h_+ - y^h_-),
\end{aligned}
$$

and the optimal contract is characterized as follows:

	$\mathbf{x}$	$\mathbf{w}_+$	$\mathbf{w}_-$	$\mathbf{c}_+$	$\mathbf{c}_-$
$\mathbf{w} \in (\mathbf{0}, \mathbf{w}^{h,l}_{\rho=r})$	$w / w^{h,l}_{\rho=r}$	$> w$	0	0	0
$\mathbf{w} \in [\mathbf{w}^{h,l}_{\rho=r}, \mathbf{w}^{h,m}_{\rho=r})$	1	$> w$	$< w$	≥ 0	0
$\mathbf{w}^{h,m}_{\rho=r}$	1	$w^{h,m}_{\rho=r}$	$w^{h,m}_{\rho=r}$	$\lambda(y^h_+ - y^h_-)$	0

$$
w_+ = \min\{(1+r)\left[\tfrac{w}{x} + (1-p)\,\lambda(y^h_+ - y^h_-)\right], w^{h,m}_{r=\rho}\},
$$
$$
w_- = (1+r)\left[w - p\lambda(y^h_+ - y^h_-)\right],
$$
$$
c_+ = \max\{w - (w^{h,m}_{\rho=r} - \lambda(y^h_+ - y^h_-)), 0\}.
$$

The Optimal Contract

- From now on assume $\rho > r$.
- Recall the constraints restricting w_+ and w_-:

$$\begin{aligned} w &\geq x\left[\frac{w_-}{1+\rho h} + p\lambda\left(y_+^h - y_-^h\right)\right], \\ w &\geq x\left[\frac{pw_+ + (1-p)\,w_-}{1+\rho h}\right]. \end{aligned}$$

- For $w \geq w^{h,m}$ both constraints are slack and it is optimal to set $w_+ = w_- = w^{h,r}$, which, assuming differentiability, satisfies

$$V^{h\prime}(w^{h,r}) = \frac{(\rho - r)\,h}{(1+\rho h)} > 0.$$

- At $w^{h,m}$ only the first constraint is just binding, implying that

$$w^{h,m} = \frac{w^{h,r}}{1+\rho h} + p\lambda\left(y_+^h - y_-^h\right).$$

The Optimal Contract

- For $w^{h,d} := w^{h,m} - \lambda \left(y_+^h - y_-^h \right) \leq w \leq w^{h,m}$ only the first constraint is binding. Thus, it is optimal to set $w_+ = w^{h,r}$ and

$$w_- = (1 + \rho h) \left[w - p\lambda \left(y_+^h - y_-^h \right) \right].$$

- For $w \in \left[w^{h,l}, w^{h,d} \right)$ both constraints are binding and

$$\begin{aligned} w_+ &= (1 + \rho h) \left[w + (1 - p) \lambda \left(y_+^h - y_-^h \right) \right], \\ w_- &= (1 + \rho h) \left[w - p\lambda \left(y_+^h - y_-^h \right) \right]. \end{aligned}$$

- Finally, for $w < w^{h,l}$ there is positive probability of termination. (As for $w^{h,m}$ there is also no closed form solution for $w^{h,l}$ in this case.)

The Optimal Contract

- Given w_+ and w_-, one can then obtain the transfers $c_{+,-}$ from the constraints of the original problem (IC) and (PK):

$$\begin{aligned} c_+ &= \max\left\{w - w^{h,d}, 0\right\}, \\ c_- &= \max\left\{w - w^{h,m}, 0\right\}. \end{aligned}$$

- By construction, $w^{h,r} < w^{h,m}$ is a reflecting boundary for w, i.e., once $w \leq w^{h,r}$ it stays smaller than $w^{h,r}$ forever.
- Intuition: As $\rho > r$ it is no longer optimal to wait till the agent's stake in the firm is large enough to reduce the probability of termination to zero.
$\longrightarrow$ Increase termination probability for earlier consumption,
$\longrightarrow$ "Immiserization":
- **Proposition 3:** *When $\rho > r$ the firm is liquidated with probability one in the long run:*

$$\lim_{n\to\infty} \prod_{j=1}^{n} x_{nh} = 0, \ a.s.$$

The Optimal Contract

Proposition 4: *The optimal contract is characterized by two regimes:*

(i) *If* $w \in [0, w^{h,l})$, *the project is continued with probability* $x = w/w^{h,l}$ *and liquidated with probability* $1 - x$. *If the project is continued, the optimal contract starting at* $w/x = w^{h,l}$ *is immediately executed.*

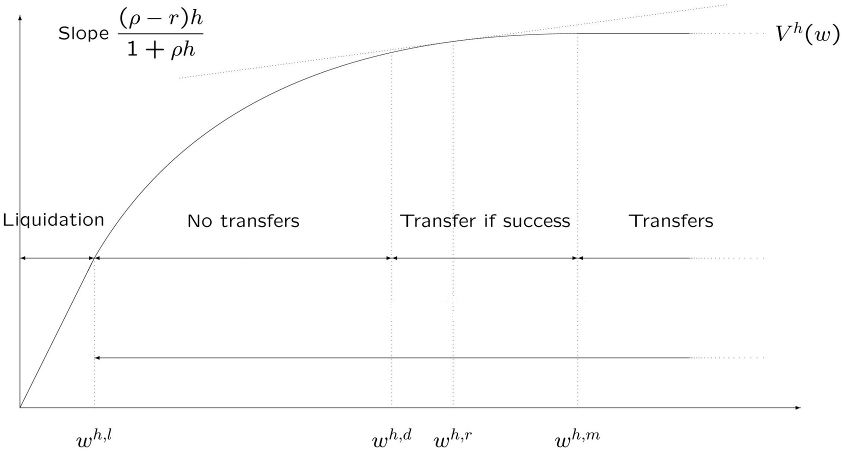
(ii) *If* $w \in [w^{h,l}, \infty)$, *the project is continued with probability* 1. *The optimal continuation utilities are given by*

$$\begin{aligned} w_+ &= \min\left\{(1+\rho h)\left[w + (1-p)\,\lambda\left(y_+^h - y_-^h\right)\right], w^{h,r}\right\}, \\ w_- &= \min\left\{(1+\rho h)\left[w - p\lambda\left(y_+^h - y_-^h\right)\right], w^{h,r}\right\}, \end{aligned}$$

while the optimal current transfers are given by

$$\begin{aligned} c_+ &= \max\left\{w - w^{h,d}, 0\right\}, \\ c_- &= \max\left\{w - w^{h,m}, 0\right\}. \end{aligned}$$

The Optimal Contract

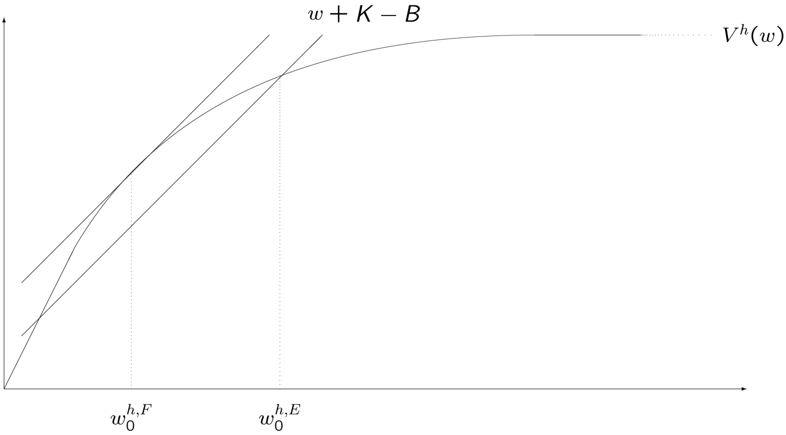


The Optimal Contract - Initialization

- From
$$V^h(w) = F(w) + w,$$

  1. If the financiers have all bargaining power they choose $w_0^{h,F}$ such that
$$\begin{aligned} & \frac{d}{dw} F^h(w_0^{h,F}) = 0 \\ \iff \quad & \frac{d}{dw} V^h(w_0^{h,F}) = 1, \end{aligned}$$
  2. If the entrepreneur has all the bargaining power, he chooses the highest $w_0^{h,E}$ such that
$$\begin{aligned} & F(w_0^{h,E}) = K - B \\ \iff \quad & V^h(w_0^{h,E}) = w_0^{h,E} + K - B. \end{aligned}$$

The Optimal Contract - Initialization



The Dynamics of w

- Recall the cash flow process

$$y_{nh}^h = \begin{cases} y_+^h = \mu h + \sigma \varepsilon_+ \sqrt{h} & \textit{with prob. } p \\ y_-^h = \mu h + \sigma \varepsilon_- \sqrt{h} & \textit{with prob. } 1-p \end{cases},$$

with $\varepsilon_+ = \sqrt{\frac{1-p}{p}}$ and $\varepsilon_- = -\sqrt{\frac{p}{1-p}}$, so

$$\begin{aligned} E\left[y_{nh}^h\right] &= \mu h, \\ Var\left[y_{nh}^h\right] &= \sigma^2 h. \end{aligned}$$

- This implies that the innovation $\varepsilon_n = \left(y_{nh}^h - E_{n-1}\left[y_{nh}^h\right]\right)/\sigma\sqrt{h}$ is a martingale difference as

$$E_{n-1}\left[\varepsilon_n\right] = 0.$$

The Dynamics of w

- Now, consider $w \in \left[w^{h,l}, w^{h,d}\right]$, i.e., no liquidation and no transfers. Then the optimal contract implies that between these thresholds it holds that

$$\begin{aligned} w^h_{(n+1)h} &= (1+\rho h)\left[w^h_{nh} + \lambda \underbrace{\left(y^h_{nh} - \mu h\right)}_{=\sigma \varepsilon_n \sqrt{h}}\right] \\ \Leftrightarrow & \quad \frac{w^h_{(n+1)h}}{(1+\rho h)} - w^h_{nh} = \lambda \sigma \varepsilon_n \sqrt{h}. \end{aligned}$$

- So w^h_{nh} is a discounted martingale and its sensitivity to the cash flow innovation is equal to λ, which measures the severity of the agency problem.

The Optimal Contract with Downsizing

- Convenient for implementation to interpret x as an irreversible downsizing factor:

  When $x < 1$, a fraction $1 - x$ of the project is liquidated.

- Assume constant returns to scale: Cash flows and utilities are scaled down by factor x.

- Both w and $v^h(w)$ are then "size adjusted".

- Downsizing decision:

  - For $w \in [w^{h,l}, \infty)$, no downsizing,
  - For $w \in [0, w^{h,l})$, firm is scaled down by $x = w/w^{h,l}$ and continuation contract starts at size adjusted continuation utility $w/x = w^{h,l}$.

Implementation

- Implementation with cash reserves, stocks and bonds (limited liability).
- All values in size adjusted terms, with firm size at beginning of period n equal to $\prod_{i=0}^{n-1} x_{ih}^{h}$.
- The firm holds cash reserves m on an account with interest rate r:

$$m_{nh}^{h} = \frac{w_{nh}^{h}}{\lambda^{h}},$$

with $\lambda^{h} = (1 + \rho h)\,\lambda / \,(1 + rh)$.

- The manager holds fraction λ of stocks, investors hold $(1 - \lambda)$ of stocks and all bonds.

Implementation

- If $m \in \left[0, w^{h,l}/\lambda^h\right)$, the firm is scaled down by $x = m/\left(w^{h,l}/\lambda^h\right)$,
- If $m \in \left[w^{h,l}/\lambda^h, w^{h,r}/\lambda^h\right]$, following a success, stocks distribute a size-adjusted dividend

$$e = \max\left\{\frac{\lambda^h m}{\lambda} - \frac{w^{h,d}}{\lambda}, 0\right\},$$

and bonds distribute a size-adjusted coupon

$$b = \mu h - \frac{(\rho - r)\, hm}{1 + rh}.$$

- Given this definitions, starting at $m_0^h = w_0^h/\lambda^h$ the cash holdings evolve according to

$$m_{(n+1)h}^h = (1 + rh)\left(m_{nh}^h + y_{nh}^h - b_{nh}^h - e_{nh}^h\right),$$

which is equivalent to the evolution of w_{nh}^h derived above.

Part 2:
The Continuous Time Limit.

Convergence of the Value Functions

- Consider again $w \in \left[w^{h,l}, w^{h,d}\right]$ such that the continuation utility of the agent evolves according to

$$w^h_{(n+1)h} = (1+\rho h)\left[w^h_{nh} + \lambda\sigma\varepsilon_n\sqrt{h}\right].$$

- Then, for $(\widetilde{w}, \widetilde{\varepsilon}) \in \{(w_+, \varepsilon_+), (w_-, \varepsilon_-)\}$ Taylor approximation yields

$$V^h(\widetilde{w}) = V^h(w) + \left(\rho h w + \lambda\sigma\widetilde{\varepsilon}\sqrt{h}\right) V^{h\prime}(w) + \frac{\lambda^2\sigma^2\widetilde{\varepsilon}^2 h}{2} V^{h\prime\prime}(w) + o(h).$$

- Substituting this into the Bellman equation, one obtains the following approximation:

$$rV^h(w) \sim \mu - (\rho - r)\, w + \rho w V^{h\prime}(w) + \frac{\lambda^2\sigma^2}{2} V^{h\prime\prime}(w).$$

Convergence of the Value Functions

Proposition 5: *As h goes to 0, the value function V^h converges uniformly to the unique solution V to the free boundary problem*

$$rV(W) = \begin{cases} \mu - (\rho - r)\,W + \rho W V'(W) + \frac{\lambda^2\sigma^2}{2} V''(W) & \text{if } W \in [0, W^m] \\ rV(W^m) & \text{if } W \in (W^m, \infty) \end{cases},$$

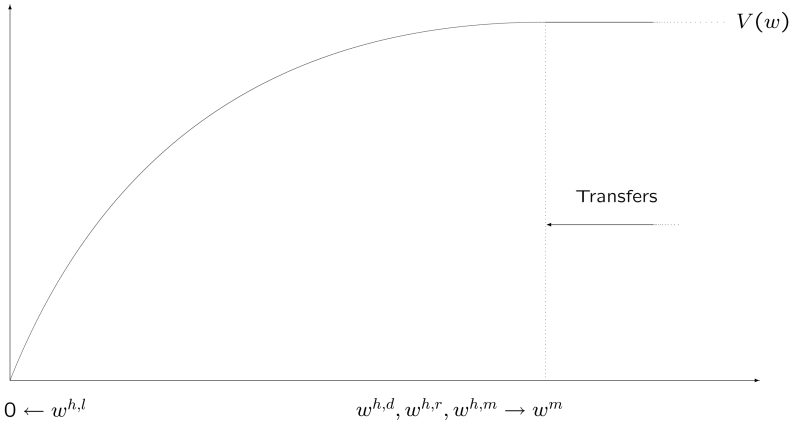
with boundary conditions

$$\begin{aligned} V(0) &= 0, \\ V'(W^m) &= 0, \\ V''(W^m) &= 0. \end{aligned}$$

It holds that

$$\begin{aligned} \lim_{h \to 0} w^{h,l} &= 0, \\ \lim_{h \to 0} w^{h,d} &= \lim_{h \to 0} w^{h,r} = \lim_{h \to 0} w^{h,m} = W^m. \end{aligned}$$

Convergence of the Value Functions



Convergence of Cash Flows

- The total revenue generated by the project up to any date nh prior to liquidation is
$$Y_{nh}^{h} = \mu\,(n+1)\,h + \sigma \sum_{i=0}^{n} \varepsilon_i \sqrt{h}.$$
- This converges for $h \to 0$ to the arithmetic Brownian motion
$$Y_t = \mu t + \sigma Z_t.$$

Convergence of the Optimal Contracts

- Recall: For $w \in \left[w^{h,l}, w^{h,d}\right]$ the agent's utility evolves with constant growth rate and volatility according to

$$w^h_{(n+1)h} = (1+\rho h)\left[w^h_{nh} + \lambda\sigma\varepsilon_n\sqrt{h}\right].$$

- Further,

$$\begin{aligned}
\lim_{h\to 0} w^{h,l} &= 0, \\
\lim_{h\to 0} w^{h,d} &= \lim_{h\to 0} w^{h,r} = \lim_{h\to 0} w^{h,m} = W^m.
\end{aligned}$$

- **Proposition 6:** *As $h \to 0$, the process w^h converges to the solution W to the reflected diffusion problem*

$$\begin{aligned}
dW_t &= \rho W_t dt + \lambda\sigma dZ_t - dC_t, \\
W_t &\leq W^m, \\
C_t &= \int_0^t 1_{\{W_s = W^m\}} dC_s,
\end{aligned}$$

for all $t \in [0,\tau]$, *where* $\tau = \inf\{t \geq 0 : \ W_t = 0\} < \infty$, a.s.

Convergence of the Optimal Contracts

Proposition 7: *Let $F(W) = V(W) - W$ denote the financiers' utility given a promised utility W for the entrepreneur in the continuous time limit of the model. Then, for any $W \in [0, W^m]$,*

$$W = E^{(W,0)}\left[\int_0^{\tau} e^{-\rho t} dC_t\right],$$

$$F(W) = E^{(W,0)}\left[\int_0^{\tau} e^{-rt}\left(\mu dt - dC_t\right)\right],$$

where $E^{(W,0)}$ is the expectation operator induced by the process (W, C) starting at $(W, 0)$.

Discussion

- Results show that continuous time analysis of cash flow diversion model in DeMarzo and Sannikov (2006) arises as the limit of a discrete time cash flow diversion model.
- Crucial: Cash Flows follow a binomial process.
- Sadzik and Stacchetti (2012) show dependence of continuous time limit on information structure in a richer setting:
  - Risk aversion,
  - Hidden action and hidden information,
  - General noise distribution.
- Finding: The continuous time solution in Sannikov (2008) emerges as the limit of discrete time solution in the pure hidden action case if the variance of the likelihood ratio of the noise distribution is equal to 1 (normal distribution).

Discussion

- Consider a discrete time analogue to Sannikov (2008).
- In each period $t = 0, \Delta, 2\Delta, \ldots$:
    - Agent chooses hidden effort a_t with effort costs $h(a_t)$,
    - Both parties observe cash flows $y_t = a_t + \varepsilon_t$,
    - Principal chooses wage $c_t \geq 0$.
- Noise term ε_t i.i.d with density $g(\varepsilon_t)$.

Discussion

- Denote the principal's value function by $F^{\Delta}(w)$. Then $F^{\Delta}(w)$ converges to the solution of the following boundary value problem

$$F(W) = \sup_{a,c} \left\{ (a-c) + (W - (u(c) - h(a)))\, F'(W) + \frac{r\theta(a,h(a))}{2} F''(W) \right\},$$

with boundary conditions

$$\begin{aligned} F(0) &= 0, \\ F(W^{gp}) &= F_0(W^{gp}) = -u^{-1}(W^{gp}), \\ F'(W^{gp}) &= F_0'(W^{gp}). \end{aligned}$$

- The term $\theta(a, h(a))$ is given by

$$\theta(a, h(a)) = \frac{(h'(a))^2}{VLR(g_{\varepsilon})},$$

where VLR denotes the variance of the likelihood ratio ("informativeness" of public signal).

Discussion

- In particular, we have

$$VLR(g_\varepsilon) = \int \left(\frac{g'(\varepsilon)}{g(\varepsilon)}\right)^2 g(\varepsilon)d\varepsilon = \int \frac{\left(g'(\varepsilon)\right)^2}{g(\varepsilon)} d\varepsilon.$$

- When ε is normally distributed, then

$$\theta(a, h(a)) = \frac{\sigma^2 \left(h'(a)\right)^2}{1},$$

establishing the equivalence to Sannikov (2008).

- When x has bounded support and density $g(\varepsilon) = 1 - |\varepsilon|$, with $|\varepsilon| \leq 1$, then

$$\theta(a, h(a)) = \frac{\sigma^2 \left(h'(a)\right)^2}{\infty},$$

and one can achieve the first best.

- Costs of incentives θ are decreasing in VLR. Thus, $F(W)$ is increasing in VLR for any W.