Dynamic Principal Agent Models: A Continuous Time Approach

Lecture II

Dynamic Financial Contracting I - The "Workhorse Model" for Finance Applications (DeMarzo and Sannikov 2006)

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Outline

1. Solve the continuous time model with a risk-neutral agent (DeMarzo Sannikov 2006).
2. Derive analytic comparative statics.
3. Capital structure implementation(s).
4. Asset pricing implications.

DeMarzo and Sannikov 2006

- Time is continuous with $t \in [0, \infty)$.
- Risk-neutral principal with discount rate r.
- Risk-neutral agent with discount rate $\rho > r$.
- Agent has limited liability and limited wealth, so principal has to cover operating losses and initial set up costs K.

DeMarzo and Sannikov 2006

- Firm produces cash flows

$$dY_t = \mu dt + \sigma dZ_t,$$

  - with constant exogenous drift rate $\mu > 0$,
  - and Z is a standard Brownian motion.

- Principal does not observe Y but only the agent's report

$$d\hat{Y}_t = (\mu - A_t)\, dt + \sigma dZ_t.$$

  - $A \geq 0$ represents the diversion of cash flow by the agent.
  - Agent enjoys benefits from diversion of λA with $\lambda \leq 1$.
  - A revelation principle-like argument implies that it is always optimal to implement truth telling: $A_t = 0$, $t \geq 0$.

The Principal's Problem

- Find the profit-maximizing full commitment contract at $t = 0$
- A contract specifies cash payments to the agent $C = \{C_t, t \geq 0\}$ and a stopping time $\tau \geq 0$ when the firm is liquidated and the receives scrap value L, to maximize the principal's profit

$$F_0 = E^{A=0}\left[\int_0^\tau e^{-rt}\left(\mu dt - dC_t\right) + e^{-r\tau}L\right],$$

- subject to delivering the agent an initial value of W_0

$$W_0 = E^{A=0}\left[\int_0^\tau e^{-\rho t} dC_t + e^{-\rho\tau}R\right],$$

- and incentive compatibility

$$W_0 \geq E^{\tilde{A}}\left[\int_0^\tau e^{-\rho t}\left(dC_t + \lambda \tilde{A}_t dt\right) + e^{-\rho\tau}R\right], \text{ given } \{\tilde{A}_t\} \geq 0.$$

5 Steps to Solve for the Optimal Contract

1. Define agent's continuation value W_t given a contract $\{C, \tau\}$ if he tells the truth

$$W_t = E^A \left[\int_t^\tau e^{-\rho(u-t)} dC_u + e^{-\rho(\tau-t)} R \,\middle|\, \mathcal{F}_t \right]. \tag{1}$$

2. Represent the evolution of W_t over time.
3. Derive the incentive compatibility constraint under which the agent reports truthfully.
4. Derive the HJB for the principal's profits $F(W)$.
5. Verification of the conjectured contract.

5 Steps to Solve for the Optimal Contract

Step 2:
Represent the evolution of W_t over time.

Represent the Evolution of W over Time

- Exercise:
  - Define t-expectation of agent's lifetime utility V_t,
  - use MRT characterize V_t derive dW_t.
- **Theorem:** *Let Z_t be a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{Q})$ and $\mathcal{F}_t$ the filtration generated by this Brownian motion. If M_t is a martingale with respect to this filtration, then there is an $\mathcal{F}_t$-adapted process Γ such that*

$$M_t = M_0 + \int_0^t \Gamma_s dZ_s, \ 0 \leq t \leq T.$$

Evolution of Agent's Continuation Value

- The agent's continuation value evolves according to

$$dW_t = \rho W_t dt - dC_t + \Gamma_t \left(d\hat{Y}_t - \mu dt \right).$$

  - Principal has to honor his promises: W has to grow at the agent's discount rate ρ.
  - W decreases with cash payments to the agent dC_t.
  - Sensitivity with respect to firm's cash flows Γ_t will be used to provide incentives.

5 Steps to Solve for the Optimal Contract

Step 3:
Derive the local incentive compatibility constraint.

Local Incentive Compatibility Constraint

- **Proposition 1.** *The truth telling contract* $\{C, \tau\}$ *is incentive compatible if and only if*
$$\Gamma_t \geq \lambda \text{ for } t \geq 0.$$
- Intuition: Assume the agent would divert cash flows $dY_t - d\hat{Y}_t > 0$
  - immediate benefit from consumption: $\lambda \left(dY_t - d\hat{Y}_t\right)$,
  - change in continuation value W_t: $-\Gamma_t \left(dY_t - d\hat{Y}_t\right)$.

Proof of Proposition 1

- The agent's expected lifetime utility for any feasible policy with $d\hat{Y}_t \leq dY_t$, is given by

$$W_0 + \int_0^{\tau} e^{-\rho t} \lambda \left(dY_t - d\hat{Y}_t \right) - \int_0^{\tau} e^{-\rho t} \Gamma_t \left(dY_t - d\hat{Y}_t \right).$$

- Sufficiency:
  If $\Gamma_t \geq \lambda$ holds, this expression is maximized by setting $d\hat{Y}_t = dY_t \ \forall t$.
- Necessity:
  Assume $\Gamma_t < \lambda$ on a set of positive measure. Then the agent could gain by setting $d\hat{Y}_t < dY_t$ on this set.

5 Steps to Solve for the Optimal Contract

Step 4:
Derivation of the HJB for the principal's value function.

Derivation of the HJB for Principal's Value Function

- Denote the highest profit that the principal can get from a contract, that provides the agent expected payoff W, by

$$F(W).$$

- Assume for now that the principal's value function is concave:

$$F''(W) \leq 0$$

  (this will be verified later).

- Principal dislikes variation W as the agent has to be fired – which is inefficient – if $W = 0$.

Optimal Compensation Policy

- Principal has two options for compensating the agent:
  - Raise agent's promised pay W at marginal costs of $F'(W)$,
  - lump sum payment to the agent at marginal costs of -1.

$\Rightarrow$ No cash payments as long as $F'(W) > -1$

- Define the compensation threshold $\overline{W}$ by

$$F'\left(\overline{W}\right) = -1,$$

- where cash payments reflect W at $\overline{W}$, i.e.

$$dC = \max\left\{0, W - \overline{W}\right\}. \tag{2}$$

Derivation of the HJB for Principal's Value Function

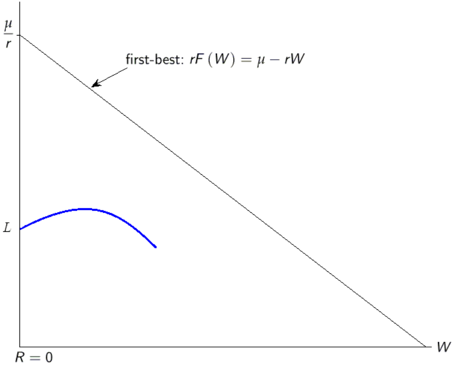
- Exercise:
  - What does the evolution of W_t look like for $W_t \in \left[R, \overline{W}\right]$?
  - Derive the HJB for $W_t \in \left[R, \overline{W}\right]$
    - What is the principal's required rate of return?
    - What is the instantaneous cash flow?
    - Use Itô's lemma, derive the differential $dF\left(W\right)$.

Boundary Conditions

- *Value matching*

$$F(R) = L$$

If the agent is fired, the principal gets liquidation value L.



Boundary Conditions

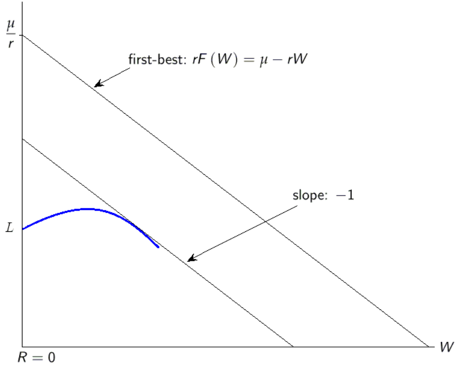
- *Value matching*

$$F(R) = L$$

- *Smooth pasting*

$$F'\left(\overline{W}\right) = -1$$

At compensation boundary marginal costs of cash payments have to match those of raising W.



Boundary Conditions

- *Value matching*

$$F(R) = L$$

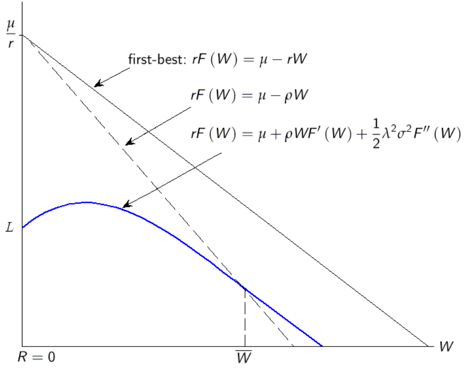
- *Smooth pasting*

$$F'\left(\overline{W}\right) = -1$$

- *Super contact*

$$F''\left(\overline{W}\right) = 0$$

Ensures optimal choice of compensation boundary $\overline{W}$.

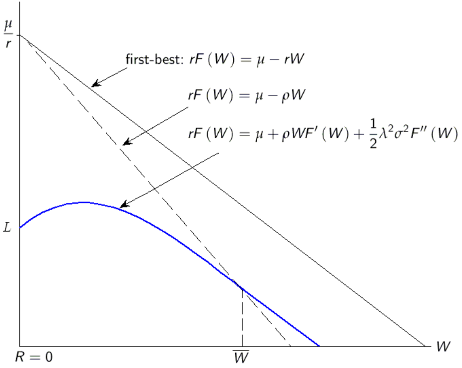


Boundary Conditions

- Concavity of F reflects following trade off:
- Raising W has ambiguous marginal effect on F

$+$ **less** risk of termination ($L < \mu / r$) (weaker for high W).

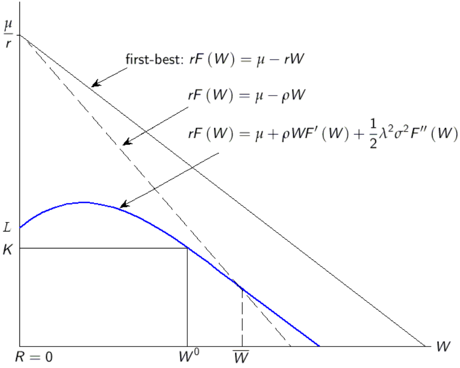
$-$ **more** cash payments in the future ($\rho > r$) (independent of W).



Relative Bargaining Power and Distribution of Surplus

- If **investors** are competitive, W_0 is the largest W such that investors break even.
- Since investors make zero profits, denote this value by W^0:

$$F\left(W^0\right) = K.$$



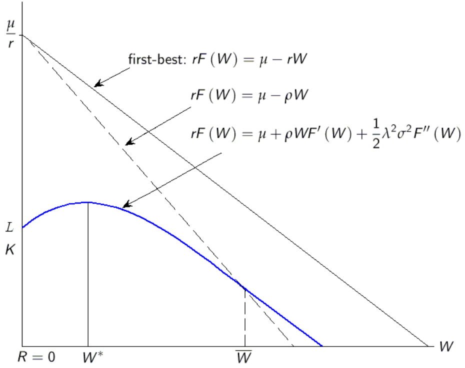
Relative Bargaining Power and Distribution of Surplus

- If **managers** are competitive, $W_0 = W^*$, where

$$W^* = \arg\max_{W} F(W).$$

- The project is funded initially only if

$$F(W^*) \geq K.$$



A Note on Commitment and Renegotiation

- It is assumed that the principal can commit to a long-term contract.
- However, the principal may want to renegotiate the contract:
  - When $F'(W) > 0$, he may not want to reduce W following a bad cash flow shock.
  - More generally: Principal and agent would benefit from raising W.
- This can be dealt with by imposing the restriction that $F(W)$ is non-increasing:
- The optimal contract will be terminated randomly at lower boundary $\underline{W} > R$:

$$dW_t = \rho W_t - dC_t + \Gamma_t \sigma dZ_t + dP_t,$$

  with P reflecting W at $\underline{W}$ and the project continued with probability $dP_t / (\underline{W} - R)$.

5 Steps to Solve for the Optimal Contract

Step 5:
Verification of the conjectured contract.

Concavity of F(W)

- **Proposition 2.** *For* $W \in [R, \overline{W})$, *it holds that* $F''(W) < 0$.
- **Proof.**

1. Use boundary conditions to show that $F''\left(\overline{W} - \varepsilon\right) < 0$:
   - Differentiating HJB w.r.t. W yields

$$(r - \rho) F'(W) = \rho W F''(W) + \frac{1}{2}\lambda^2\sigma^2 F'''(W). \tag{3}$$

   - Evaluating (3) in $\overline{W}$ implies

$$F'''(\overline{W}) = 2\frac{\rho - r}{\lambda^2\sigma^2} > 0,$$

   - from which we get $F''(\overline{W} - \varepsilon) < 0$.

Proof of Proposition 2

2. Assume that there is a $\tilde{W}$ s.t.

$$\begin{aligned} F''\left(\tilde{W}\right) &= 0 \text{ and} \\ F''\left(W\right) &< 0 \text{ for } W \in (\tilde{W}, \overline{W}). \end{aligned}$$

- By continuity, this implies that $F'''\left(\tilde{W}\right) < 0$ and, from (3),

$$F'(\tilde{W}) = -\frac{1}{2}\frac{\lambda^2\sigma^2}{\rho - r}F'''(\tilde{W}) > 0. \tag{4}$$

- The joint surplus has to be strictly lower than first best:

$$F\left(W\right) + W < \frac{\mu}{r},$$

so that, from evaluating the HJB in $\tilde{W}$, we would get

$$F\left(\tilde{W}\right) + \tilde{W} - \frac{\mu}{r} = \tilde{W} + \frac{\rho}{r}\tilde{W}F'\left(\tilde{W}\right),$$

implying that $F'\left(\tilde{W}\right) < 0$, contradicting (4).

Verification Theorem

- We still need to verify that the principal's profits are maximized under the conjectured contract.
- Define the principal's lifetime profits for any incentive compatible contract:

$$G_t = \int_0^t e^{-rs}\left(dY_s - dC_s\right) + e^{-rt} F\left(W_t\right),$$

- and look at the drift of G (use Itô's Lemma and dynamic of W)

$$\underbrace{\left[\mu + \rho F'\left(W\right) + \frac{1}{2}\Gamma_t^2 \sigma^2 F''\left(W\right) - rF\left(W_t\right)\right]}_{\leq 0} dt - \underbrace{\left[1 + F'\left(W\right)\right]}_{\geq 0} dC_t.$$

- The first statement holds with equality under the conjectured contract, that is if the HJB is satisfied.
- The second statement holds with equality if dC follows (2): $dC > 0$ only if $F'\left(W\right) = -1$.

Verification Theorem

- Therefore G is a supermartingale and a martingale under the conjectured contract

$\Rightarrow$ $F\left(W\right)$ provides an upper bound of the principal's profits under any incentive compatible contract, as

$$E\left[\int_0^\tau e^{-rt}\left(dY_t - dC_t\right) + e^{-r\tau}L\right] = E\left[G_\tau\right] \leq G_0 = F\left(W_0\right),$$

with equality under the optimal contract.

Comparative Statics

Derive analytical comparative statics.

Comparative Statics

- Discrete time: Comparative statics often analytically intractable.
- Continuous time: Characterization of optimal contract with ODE allows for analytical comp. statics.
- Effect of a particular parameter θ on value function $F_\theta(W)$ can be found as follows:
    1. Differentiate the HJB and its boundary conditions with respect to θ, keeping $\overline{W}$ fixed (envelope theorem) giving a 2^{nd} order ODE in $\partial F_\theta(W)/\partial\theta$ with appropriate boundary conditions.
    2. Apply a Feynman-Kac style argument to write the solution as an expectation, which can be signed in many cases.

Comparative Statics

- Given $\overline{W}$ the principal's profit function $F_{\theta,\overline{W}}(W)$ solves the following boundary value problem

$$
\begin{aligned}
rF_{\theta,\overline{W}}(W) &= \mu + \rho W F'_{\theta,\overline{W}}(W) + \frac{1}{2}\lambda^2\sigma^2 F''_{\theta,\overline{W}}(W), \\
F_{\theta,\overline{W}}(R) &= L, \; F'_{\theta,\overline{W}}(\overline{W}) = -1.
\end{aligned}
$$

- Differentiating with respect to θ and evaluating at the profit maximizing choice $\overline{W} = \overline{W}(\theta)$, gives

$$
\begin{aligned}
r\frac{\partial F_\theta(W)}{\partial\theta} &= \frac{\partial\mu}{\partial\theta} + \frac{\partial\rho}{\partial\theta} W F'_\theta(W) + \rho W \frac{\partial}{\partial W}\frac{\partial F_\theta(W)}{\partial\theta} \\
&\quad + \frac{1}{2}\frac{\partial\lambda^2\sigma^2}{\partial\theta} F''_\theta(W) + \frac{1}{2}\lambda^2\sigma^2 \frac{\partial^2}{\partial W^2}\frac{\partial F_\theta(W)}{\partial\theta}
\end{aligned}
$$

with boundary conditions

$$
\frac{\partial F_\theta(R)}{\partial\theta} = \frac{\partial L}{\partial\theta}, \; \frac{\partial}{\partial W}\frac{\partial F_\theta\left(\overline{W}\right)}{\partial\theta} = 0,
$$

where we have used the envelope theorem $\partial F_\theta(W)/\partial\theta = \partial F_{\theta,\overline{W}(\theta)}(W)/\partial\theta$.

Comparative Statics

- For notational simplicity, write $G(W) := \partial F_\theta(W)/\partial\theta$, so that

$$
\begin{aligned}
rG(W) &= \underbrace{\frac{\partial\mu}{\partial\theta} + \frac{\partial\rho}{\partial\theta} W F_\theta'(W) + \frac{1}{2}\frac{\partial\lambda^2\sigma^2}{\partial\theta} F_\theta''(W)}_{=:g(W)} \\
&\quad + \rho W G'(W) + \frac{1}{2}\lambda^2\sigma^2 G''(W), \\
G(R) &= \frac{\partial L}{\partial\theta},\ \ G'(\overline{W}) = 0.
\end{aligned}
$$

- "Find the martingale": Next, define

$$
H_t = \int_0^t e^{-rs} g(W_s)\,ds + e^{-rt} G(W_t).
$$

Comparative Statics

- From Itô's lemma

$$
\begin{aligned}
e^{rt} dH_t &= \left(g(W_t) + \rho W_t G'(W_t) + \frac{1}{2} G''(W_t) \lambda^2 \sigma^2 - rG(W_t) \right) dt \\
&\quad - G'(W_t) dI_t + G'(W_t) \lambda \sigma dZ_t,
\end{aligned}
$$

showing that H_t is a martingale:

$$
G(W_0) = H_0 = E[H_\tau] = E\left[\int_0^\tau e^{-rt} g(W_t) dt + e^{-r\tau} \frac{\partial L}{\partial \theta} \right].
$$

- Plugging back the definition of $G(W)$:

$$
\begin{aligned}
&\frac{\partial F_\theta(W)}{\partial \theta} \\
= \quad & E\left[\begin{array}{c} \int_0^\tau e^{-rt} \left(\frac{\partial \mu}{\partial \theta} + \frac{\partial \rho}{\partial \theta} W_t F'_\theta(W_t) + \frac{1}{2} \frac{\partial \lambda^2 \sigma^2}{\partial \theta} F''_\theta(W_t) \right) dt \\ + e^{-r\tau} \frac{\partial L}{\partial \theta} \end{array} \middle| W_0 = W \right].
\end{aligned}
$$

Comparative Statics

- For comparative statics with respect to R note that the principal's profit remains unchanged if the agent's outside option increases by dR and the liquidation value increases by $F'(R)dR$, hence:

$$\frac{\partial F(W)}{\partial R} = -F'(R)E\left[\left. e^{-r\tau} \right| W_0 = W\right].$$

- Given the effect of θ on $F_\theta(W)$ we get:
  - the change in $\overline{W}$ from $rF_\theta\left(\overline{W}\right) + \rho\overline{W} = \mu$,
  - the change in W^* from $F'(W^*) = 0$,
  - the change in W^0 from $F(W^0) = K$.

Comparative Statics

- Example:
$$\frac{\partial F(W)}{\partial L} = E\left[\left. e^{-r\tau} \right| W_0 = W\right] > 0,$$
  - from $rF\left(\overline{W}\right) + \rho\overline{W} - \mu = 0$ one gets
$$\frac{\partial \overline{W}}{\partial L} = -\frac{rE\left[\left. e^{-r\tau} \right| W_0 = \overline{W}\right]}{\rho - r} < 0,$$
  - from $F'(W^*) = 0$ it holds that
$$\frac{\partial W^*}{\partial L} = -\frac{\frac{\partial}{\partial W} E\left[\left. e^{-r\tau} \right| W_0 = W^*\right]}{F''(W^*)} < 0,$$
  - from $F(W^0) = K$ one gets
$$\frac{\partial W^0}{\partial L} = -\frac{E\left[\left. e^{-r\tau} \right| W_0 = W^0\right]}{F'(W^0)} > 0.$$

Capital Structure Implementation

The optimal contract can be implemented using standard securities.

Capital Structure Implementation

- Equity
  - Equity holders receive dividend payments.
  - Dividend payments are made at agent's discretion.
- Long-term Debt
  - Console bond that pays continuous coupons.
  - If firm defaults on a coupon payment, debt holders force termination.
- Credit Line
  - Revolving credit line with limit $\overline{W}$.
  - Drawing down and repaying credit line is at the agent's discretion.
  - If balance on the credit line M_t exceeds $\overline{W}$, firm defaults and is liquidated (creditors receive L).

Capital Structure Implementation

- Agent has no incentives to divert cash flows if he is entitled to fraction λ of the firm's equity and has discretion over dividend payments. (For simplicity, take $\lambda = 1$ for now.)
- Idea: Construct a capital structure that allows to use the balance on credit line M_t as "memory device" in lieu of the original state variable W_t:

$$M_t = \overline{W} - W_t.$$

- To keep the balance M positive, dividends have to be distributed once credit line is fully repaid ($M_t = 0$).
- Firm is liquidated when credit line is overdrawn ($M_t = \overline{W}$).

Capital Structure Implementation

- To implement our optimal contract, the balance on the credit line has to mirror the agent's continuation value W_t. Hence, $M_t = \overline{W} - W_t$ follows

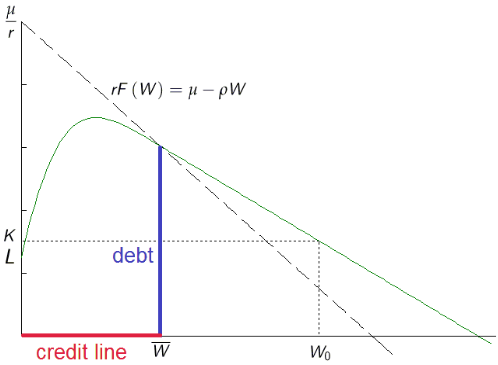
$$dM_t = \underbrace{\rho M_t dt}_{\text{interest on c.l.}} + \underbrace{\left(\mu - \rho\overline{W}\right) dt}_{\text{coupon payment}} + \underbrace{dC_t}_{\text{dividend}} - \underbrace{d\hat{Y}_t}_{\text{cash flow}} .$$

  - Credit line charges an interest rate equal to agent's discount rate ρ.
  - Letting coupon rate be r, face value of long-term debt is equal to

$$D = \frac{\mu}{r} - \frac{\rho}{r}\overline{W} = F\left(\overline{W}\right).$$

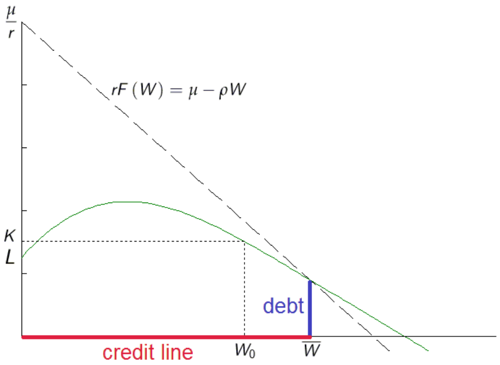
  - Dividend payments are paid out of credit line.
  - Cash inflows are used to pay back the credit line.

Capital Structure – Low Risk



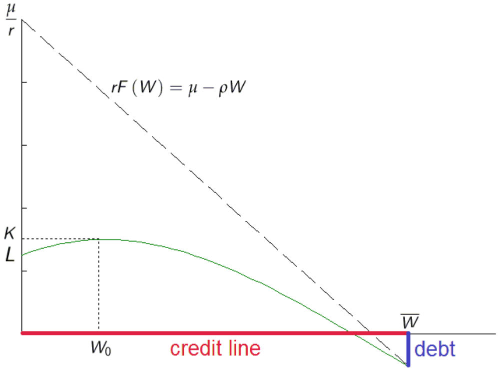
- Debt is risky, as $D > L$ and must trade at a discount.
- Lenders expect to earn a profit from credit line (charging high interest ρ), which exactly offsets this discount.

Capital Structure – Intermediate Risk



- Higher risk calls for a longer credit line (financial slack) and a lower level of debt (debt is now riskless, as $D < L$).
- Difference in set up costs $K - D$ is financed by initial draw on credit line $\overline{W} - W_0$, for which lenders charge a "fee" of $(\overline{W} - W_0) - (K - D) > 0$.

Capital Structure – High Risk



- Negative debt: cash deposit as condition for extremely long credit line.
- Interest earned on D increases profitability of firm to deter agent from consuming credit line and defaulting.

Comparative Statics for the Implementation

	dC^L	dD	dW^*	dW^0	$dF(W^*)$
dL	$-$	$+$	$-$	$+$	$+$
dR	$-$	$-$	$+$	$-$	$-$
$d\rho$	$-$	$\pm$	$-$	$-$	$-$
$d\mu$	$+$	$\pm$	$+$	$+$	$+$
$d\sigma^2$	$+$	$-$	$\pm$	$-$	$-$
$d\lambda$	$\pm$	$+$	$\pm$	$-$	$-$

- Credit line decreases in L as financial slack is less valuable.
- Credit line decreases in ρ as it becomes costlier to delay compensation.
- Credit line increases in μ, σ^2 to reduce probability of termination.

Comparative Statics for the Implementation

	dC^L	dD	dW^*	dW^0	$dF(W^*)$
dL	$-$	$+$	$-$	$+$	$+$
dR	$-$	$-$	$+$	$-$	$-$
$d\rho$	$-$	$\pm$	$-$	$-$	$-$
$d\mu$	$+$	$\pm$	$+$	$+$	$+$
$d\sigma^2$	$+$	$-$	$\pm$	$-$	$-$
$d\lambda$	$\pm$	$+$	$\pm$	$-$	$-$

- Firm becomes more profitable as L and μ increase.
- Firm becomes less profitable as R, ρ, σ and λ increase.

Capital Structure Implementation II

Security Pricing

Security Prices

- There is more we can say about security prices. Consider an alternative implementation, where $\tilde{M} = W/\lambda$ denotes the firm's cash reserves (this follows Biais et al. 2007)

$$d\tilde{M}_t = \rho \tilde{M}_t dt + \sigma dZ_t - \frac{1}{\lambda} dC_t.$$

- The firm is liquidated if its cash reserves are exhausted ($W_t/\lambda = 0$),
- the agent distributes a dividend dC_t/λ when cash reserves meet an upper bound $\overline{W}/\lambda$.
- Rewrite the evolution of $\tilde{M}$

$$d\tilde{M}_t = r(\tilde{M}_t + \mu)dt + \sigma dZ_t - dC_t - dP_t,$$

where dC_t denotes the agent's fraction of dividends and dP_t payments to bond holders and holders of external equity, respectively, with

$$dP_t = \left[\mu - (\rho - r)\,\tilde{M}_t\right] dt + \frac{1-\lambda}{\lambda} dC_t.$$

Stock Price

- The market value of stocks is equal to expected dividend payments

$$S_t = E_t\left[\int_t^{\tau} e^{-r(s-t)}\frac{1}{\lambda}dC_s\right].$$

- By Itô's formula, $S\left(\tilde{M}\right)$ has to satisfy the following differential equation over $\tilde{M} \in [0, \overline{W}/\lambda]$

$$rS\left(\tilde{M}\right) = \rho\tilde{M}S'(\tilde{M}) + \frac{1}{2}\sigma^2 S''(\tilde{M}).$$

with boundary conditions

$$\begin{aligned} S\left(0\right) &= 0, \\ S'\left(\frac{\overline{W}}{\lambda}\right) &= 1. \end{aligned}$$

Stock Price (Testable Implications)

- *Stock price $S\left(\tilde{M}\right)$ is (a) increasing and (b) concave in cash holdings $\tilde{M}$.*
- Intuition:
  (a) An increase in cash holdings $\tilde{M}$ reduces probability of default and increases probability of dividend payment.
  (b) For low $\tilde{M}$, threat of default is more immediate $\Rightarrow$ Stock price reacts more strongly to firm performance when cash holdings are low.

Stock Price (Testable Implications)

- From Itô's formula, the stock price follows

$$dS_t = rS_t dt + S_t \sigma^S(S_t)\, dZ_t - \frac{1}{\lambda} dC_t,$$

where the volatility of S is given by

$$\sigma^S(s) = \frac{\sigma S'\left(S^{-1}(s)\right)}{s}.$$

- Differences to "standard" asset pricing models:
  - Stock price is reflected when dividends are paid at $S\left(\overline{W}/\lambda\right)$,
  - the volatility of the stock price remains strictly positive when $S \to 0$

$$S\sigma^S(S) = \sigma S'(\tilde{M}) > 0.$$

  - Because $S\sigma^S(S)$ is decreasing in S, the stock price is negatively correlated with its volatility "Leverage effect".

Value of Bonds

- The market value of bonds is equal to expected coupon payments

$$D_t = E_t\left[\int_t^{\tau} e^{-r(s-t)}\left[\mu - (\rho - r)\,\tilde{M}_s\right] ds\right]$$

- By Itô's formula, $D\left(\tilde{M}\right)$ has to satisfy

$$rD\left(\tilde{M}\right) = \mu - (\rho - r)\,\tilde{M}_s + \rho \tilde{M} D'(\tilde{M}) + \frac{1}{2}\sigma^2 D''(\tilde{M})$$

over $\tilde{M} \in [0, \overline{W}/\lambda]$ with boundary conditions

$$\begin{aligned} D(0) &= 0, \\ \text{and } D'\left(\frac{\overline{W}}{\lambda}\right) &= 0. \end{aligned}$$

Leverage (Testable Implications)

- *The leverage ratio D_t / S_t is strictly decreasing in $\tilde{M}_t$ and S_t.*
- Intuition:
  - Debt value reacts less to firm performance than stock price because coupon is paid steadily as long as firm operates.
  - Dividend payments on the other hand are only made after sufficiently positive record and thus react more strongly to firm performance.
- *Performance (cash flow) shocks induce persistent changes in capital structure.*
  - Puzzling in context of (static) trade-off theory: Why do firms not issue or repurchase debt/equity to restore optimal capital structure? (Welch 2004).
  - Under our dynamic contract, financial structure is adjusted optimally by change in market values of debt and equity.

Default Risk (Testable Implications)

- As a measure for the risk of default at time t, define the credit yield spread Δ_t by

$$\int_t^{\infty} e^{-(r+\Delta_t)(s-t)} ds = E_t\left[\int_t^{\tau} e^{-r(s-t)} ds\right],$$

- from which we get

$$\Delta_t = r\frac{T_t}{1-T_t},$$

where $T_t = E_t\left[e^{-r(\tau-t)}\right]$ denotes the t-expected value of one unit paid at the time of default.

Default Risk (Testable Implications)

- The credit yield spread is (a) decreasing and (b) convex in $\tilde{M}_t$.
- Intuition:

  (a) Higher cash reserves reduce the probability of default,
  (b) effect weaker for high values of $\tilde{M}_t$: At $\overline{W}/\lambda$, inflows are paid out as dividend and do not affect default risk.