Dynamic Principal Agent Models: A Continuous Time Approach

Lecture I

The "Standard" Continuous Time Principal Agent Model (Sannikov 2008)

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Outline

- Part 1: A refresher of dynamic agency in discrete time.
  - Introduce simple repeated moral hazard model,
  - Show core results from discrete time models.
- Part 2: The continuous time approach.
  - Set-up of the basic principal agent model in continuous time.
  - Outline of core steps to derive the optimal contract in (class of) continuous time models.
  - Discussion of techniques used to derive the optimal contract.
  - Discussion of properties of the optimal contract.

Part 1:

A "Refresher" of Dynamic Agency in Discrete Time.

Basic Discrete Time Theory

- Model setup:
  - Agent takes hidden action in time periods $1, 2, 3, \ldots$
  - Output depends on agent's hidden action.
  - Principal observes output and can commit to a **long-term contract** that specifies payments to the agent as a function of output history.
- Main findings:
  - Optimal contract is **history dependent** (Rogerson 1985),
  - With infinite horizon there exists a stationary representation with agent's **promised utility as state variable** (Spear and Srivastava 1987),
  - Efficiency is attainable if agent becomes patient (Radner 1985).

Basic Discrete Time Theory

Simple two period model $t = 1, 2$:

- Risk-neutral principal and risk-averse agent with common discount rate r.
- Agent's period utility is given by

$$u(C_t) - h(A_t),$$

  where A_t denotes effort and C_t denotes monetary compensation (assume that the agent cannot save/borrow).
- For simplicity assume that $A_t \in \{0, 1\}$ and $h(1) =: h$, $h(0) = 0$. Normalize $u(0) = 0$.
- Output:

$$Y_t = \begin{cases} Y^+ & \textit{with prob. } \pi(A_t) \\ Y^- & \textit{with prob. } 1 - \pi(A_t) \end{cases},$$

  where we denote $\pi(1) =: \pi$ and $\pi(0) =: \pi - \Delta\pi$, $\Delta\pi > 0$.

Basic Discrete Time Theory

- Assume that the principal wants to implement high effort in both periods.
- A contract C specifies $2+2^2$ transfers contingent on output:
  - period 1 compensation $C_1^i = C(Y_1 = Y^i)$, $i \in \{+,-\}$,
  - period 2 compensation $C_2^{i,j} = C(Y_1 = Y^i, Y_2 = Y^j)$, $i,j \in \{+,-\}$.
- This can be rewritten in terms of contingent utilities:

$$\begin{aligned} u_1^i &= u(C_1^i),\ i \in \{+,-\}, \\ u_2^{i,j} &= u(C_2^{i,j}),\ i,j \in \{+,-\}. \end{aligned}$$

Basic Discrete Time Theory

- Incentive compatibility in $t = 2$ requires:

$$u_2^{i,+} - u_2^{i,-} \geq \frac{h}{\Delta\pi}, \ i \in \{+,-\}.$$

- Denote the expected net utility from $t = 2$ conditional on Y_1 by

$$W_2^i = \pi u_2^{i,+} + (1-\pi) u_2^{i,-} - h, \ i \in \{+,-\},$$

which is called the agent's **continuation value** or **promised wealth**.

- Incentive compatibility in $t = 1$ then requires:

$$u_1^+ + \frac{1}{1+r} W_2^+ - \left(u_1^- + \frac{1}{1+r} W_2^- \right) \geq \frac{h}{\Delta\pi},$$

$\rightarrow$ Continuation utilities affect $t = 1$ incentives.

$\rightarrow$ Given W_2^i, $t = 1$ incentives are unaffected by $u_2^{i,+}$ and $u_2^{i,-}$.

Basic Discrete Time Theory

- Further, we have the $t = 1$ participation constraint:

$$W_1 = \pi \left[u_1^+ + \frac{1}{1+r} W_2^+ \right] + (1-\pi) \left[u_1^- + \frac{1}{1+r} W_2^- \right] \geq h.$$

  $\rightarrow$ Continuation utilities affect $t = 1$ participation decision.
  $\rightarrow$ Given W_2^i, $t = 1$ participation is unaffected by $u_2^{i,+}$ and $u_2^{i,-}$.

- Solve the problem backwards:

  1. For each W_2^i solve the second period problem,
  2. Given the optimal continuation contract, solve the first period problem.

Basic Discrete Time Theory

- Proceeding in this manner one obtains:

$$\begin{aligned}\frac{1}{u'(C_1^i)} &= \pi_1 \frac{1}{u'(C_2^{i,+})} + (1-\pi_1)\frac{1}{u'(C_2^{i,-})} \\ &= E\left[\left.\frac{1}{u'(C_2^{i,j})}\right| Y_1 = Y^i\right],\ i \in \{+,-\},\end{aligned}$$

$\rightarrow$ **"Inverse Euler Equation"**: Agent's inverse marginal utility is a martingale.

$\rightarrow$ Providing incentives vs. smoothing consumption.

- **Proof:** Consider an optimal incentive compatible contract C.
- Construct a new contract $\widetilde{C}$ that differs from C only following first period realization $Y_1 = Y^+$:

$$\begin{aligned}\widetilde{u}_1^+ &= u_1^+ - x, \\ \widetilde{u}_2^{+,j} &= u_2^{+,j} + (1+r)x,\ j \in \{+,-\}.\end{aligned}$$

Basic Discrete Time Theory

- Note that the new contract still induces high effort:
  - Trivial following $Y_1 = Y^-$ as $\widetilde{u}_2^{-,j} = u_2^{-,j}$, $j \in \{+,-\}$,
  - Following $Y_1 = Y^+$ high effort still optimal as $(1+r)\,x$ is constant across outcomes $\widetilde{u}_2^{+,+} - \widetilde{u}_2^{+,-} = u_2^{+,+} - u_2^{+,-}$,
  - Effort in $t = 1$ is still optimal, as for $i \in \{+,-\}$

$$
\begin{aligned}
& \widetilde{u}_1^i + \frac{1}{1+r}\left(\pi \widetilde{u}_2^{i,+} + (1-\pi)\,\widetilde{u}_2^{i,-}\right) \\
= \quad & u_1^i + \frac{1}{1+r}\left(\pi u_2^{i,+} + (1-\pi)\,u_2^{i,-}\right).
\end{aligned}
$$

  - Participation still optimal as $\widetilde{W}_1 = W_1$.
- So for $x = 0$ to be optimal, it must minimize expected payments to the agent

$$
u^{-1}(u_1^+ - x) + \frac{1}{1+r}\left(\begin{array}{c} \pi u^{-1}(u_2^{+,+} + (1+r)\,x) \\ +\,(1-\pi)\,u^{-1}(u_2^{+,-} + (1+r)\,x) \end{array}\right). \blacksquare
$$

Basic Discrete Time Theory

- The inverse Euler equation implies that the optimal contract with full commitment exhibits **memory**:
  - I.e., $t = 1$ outcome affects transfers both in $t = 1$ and in $t = 2$,
  - or: Transfers in both $t = 1$ and $t = 2$ are used to provide incentives in $t = 1$,
  - in particular: $C_1^+ > C_1^-$ and $W_2^+ > W_2^-$.
- **Proof:** Suppose by contradiction that $C_2^{+,+} = C_2^{-,+}$ and $C_2^{+,-} = C_2^{-,-}$, then

$$\begin{aligned}
\frac{1}{u'(C_1^+)} &= \pi_1 \frac{1}{u'(C_2^{+,+})} + (1-\pi_1)\frac{1}{u'(C_2^{+,-})} \\
&= \pi_1 \frac{1}{u'(C_2^{-,+})} + (1-\pi_1)\frac{1}{u'(C_2^{-,-})} = \frac{1}{u'(C_1^-)},
\end{aligned}$$

violating the incentive constraint in $t = 1$.■

Basic Discrete Time Theory

- The inverse Euler equation implies that the optimal contract tries to "front-load" the agent's consumption:
  - Intuitively: Keeping continuation utility low ensures a high marginal utility of consumption in $t = 2$ (incentives),
  - If the agent had access to savings, he would save a strictly positive amount.
- **Proof:**

$$u'(C_1^i) = \frac{1}{E\left[\frac{1}{u'(C_2^{i,j})}\middle| Y_1 = Y^i\right]} < E\left[u'(C_2^{i,j})\middle| Y_1 = Y^i\right]$$

by Jensen's inequality, showing that $u'(C)$ is a submartingale.■

Basic Discrete Time Theory

- In the infinitely repeated relationship the optimal contract exhibits a **Markov property**:
  - There exists a stationary representation with agent's continuation utility as state variable:

$$W_t = E_t\left[\sum_{k=0}^{\infty} \frac{u(C_{t+k}) - h}{(1+r)^k}\right].$$

- Intuition:
  - Agent's incentives are unchanged if we replace the continuation contract that follows a given history with a different contract that has the same continuation value.
  - Thus, to maximize the principal's profit after any history, the continuation contract must be optimal given W.

Basic Discrete Time Theory

- Given W, the optimal contract is then computed recursively:

$$F(W) = \max_{\substack{u^+, u^-, \\ W^+, W^-}} \left\{ \begin{array}{c} \pi \left(Y^+ - u^{-1}(u^+) \right) + (1-\pi) \left(Y^- - u^{-1}(u^-) \right) \\ + \frac{1}{1+r} \left[\pi F(W^+) + (1-\pi) F(W^-) \right] \end{array} \right\},$$

subject to

$$\begin{aligned} \pi \left(u^+ + \frac{1}{1+r} W^+ \right) - (1-\pi) \left(u^- + \frac{1}{1+r} W^- \right) &= W, \\ u^+ + \frac{1}{1+r} W^+ - \left(u^- + \frac{1}{1+r} W^- \right) &\geq \frac{h}{\Delta\pi}. \end{aligned}$$

Basic Discrete Time Theory

- Much of the literature with infinitely many periods has focussed on approximation results of the first-best with simple contracts under no or almost no discounting:
  - As $r \rightarrow 0$ the principal's per period expected profit converges towards its first-best value.
- Intuition:
  - Sample many observations, reward when "review" positive, punish else:
    $\rightarrow$ Inference effect.
  - Risk averse agent subject to many i.i.d. risks over time:
    $\rightarrow$ By spreading rewards and punishments over time agent becomes "perfectly diversified".

Basic Discrete Time Theory

Takeaway:

- In a dynamic model, incentives can be provided not only with current but also with promise of future payments (deferred compensation):
  - increase expected future payments after good results ("carrot"),
  - decrease expected future payments after bad results ("stick").

  $\longrightarrow$ The optimal contract is **history dependent**:

  $\longrightarrow$ Better intertemporal risk sharing, statistical inference and punishment options.
- With infinite horizon there exists a stationary representation with agent's **continuation utility** as state variable.

Part 2:
The Continuous Time Approach.

The Setting

- Time is continuous with $t \in [0, \infty)$.
- Risk-neutral principal and risk-averse agent with common discount rate r.
- Agent puts effort $A = \left\{A_t \in \left[0, \overline{A}\right], 0 \leq t < \infty\right\}$.
- Principal does not observe effort but only output:

$$dY_t = A_t dt + \sigma dZ_t,$$

  where $Z = \{Z_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{Q})$.
- Agent receives consumption $C = \{C_t \geq 0, 0 \leq t < \infty\}$, based on principal's observation of output.

The Setting

- Effort costs $h(a)$, continuous, increasing and convex, with $h(0) = 0$ and $h'(0) > 0$.
- Utility of consumption $u(c)$, continuous, increasing and concave, with $u(0) = 0$ and $\underset{c \to \infty}{lim} \; u'(c) \to 0$.
  - $\to$ **Income effect**: As agent's income increases, it becomes costlier to compensate him for effort.
  - $\to$ Agent can always guarantee himself a **non-negative net utility** by putting zero effort.

The Setting

- Some crucial assumptions:
  - Principal can commit to long-term contract,
  - Agent cannot (privately) save or borrow.
- Assumptions to be relaxed later:
  - Principal and agent tied together forever:
    $\rightarrow$ Introduce valuable outside option for agent,
    $\rightarrow$ Allow principal to replace agent at some costs.
  - Career path $\rightarrow$ promotion.

The Principal's Problem

- Focus on profit-maximizing full commitment contract at $t = 0$.
- An incentive compatible contract specifies consumption stream C and (recommended) effort A to maximize principal's (average) profit

$$E^A \left[r \int_0^\infty e^{-rt} \left(A_t - C_t \right) dt \right],$$

- subject to delivering the agent an initial (average) utility of W_0

$$W_0 = E^A \left[r \int_0^\infty e^{-rt} \left(u(C_t) - h(A_t) \right) dt \right], \text{ given effort } A,$$

- and incentive compatibility

$$W_0 \geq E^{\widetilde{A}} \left[r \int_0^\infty e^{-rt} \left(u(C_t) - h(\tilde{A}_t) \right) dt \right], \text{ given any effort } \tilde{A}.$$

The Principal's Problem

- This is a difficult problem:
  - Large space of possible contracts (history dependence),
  - Complexity of incentive constraint:
    Agent also solves a dynamic optimization problem,
    $\rightarrow$ Two dynamic optimization problems embedded in one another.
- However, it is possible to reduce the problem to an optimal stochastic control problem with agent's continuation value as state variable and with appropriate (local) incentive compatibility conditions.

5 Steps to Solve for the Optimal Contract

1. Define agent's continuation value $\{W_t, 0 \leq t < \infty\}$ for any C and A.
2. Using the Martingale Representation Theorem (MRT) derive the dynamics of W_t.
3. Necessary and sufficient conditions for the agent's effort level to be optimal (local incentive compatibility).
4. Using a Hamilton Jacobi Bellman (HJB) equation, conjecture an optimal contract.
5. Verify that the conjectured contract maximizes the principal's profit.

5 Steps to Solve for the Optimal Contract

Step 1:
Define agent's continuation value $\{W_t, 0 \leq t < \infty\}$ for any C and A.

The Agent's Continuation Value - Definition

- In a dynamic model, incentives can be provided not only with current but also with promise of future payments (deferred compensation):
  - increase expected future payments after good results ("carrot"),
  - decrease expected future payments after bad results ("stick").

  $\rightarrow$ The optimal contract is **history dependent**.
- The agent's **continuation value** keeps track of accumulated promises and is defined as the agent's total future expected utility W_t:

$$W_t(C, A) = E^A \left[r \int_t^\infty e^{-r(s-t)} \left(u(C_s) - h(A_s) \right) ds \middle| \mathcal{F}_t \right].$$

- W_t completely summarizes the past history and will serve as the unique state descriptor in the optimal contract (cf. Spear and Srivastava 1987).
- Intuitively: Agent's incentives are unchanged if continuation contract after a given history is replaced with a different contract that has the same continuation value.

The Agent's Continuation Value

- Optimal contract specifies as a function of W:
    1. Agent's consumption $\rightarrow c(W)$,
    2. Agent's (recommended) effort level $\rightarrow a(W)$,
    3. How W itself changes with the realization of output $\rightarrow$ Law of motion of W_t driven by Y_t ("pay for performance").
- Payments, recommended effort and the law of motion must be consistent, in the sense that W_t is the agent's true continuation value (**"promise keeping"**).
- It must be optimal for the agent to choose recommended effort level (**"incentive compatibility"**).

5 Steps to Solve for the Optimal Contract

Step 2:
Using the Martingale Representation Theorem (MRT) derive the dynamics of W_t.

The Agent's Continuation Value - Dynamics

- **Proposition 1:** *For any* (C, A), W_t *is the agent's continuation value if and only if*

$$dW_t = r\left(W_t - u(C_t) + h(A_t)\right)dt + r\Gamma_t\underbrace{(dY_t - A_t dt)}_{=\sigma dZ_t^A},$$

for some $\mathcal{F}_t$*-adapted process* Γ *and* $\lim_{s\to\infty} E_t\left[e^{-rs}W_{t+s}\right] = 0$.

- Intuition: Continuation value W_t
  - grows at discount rate and falls with flow of (net) utility ("promise keeping", "consistency"),
  - responds to output innovation according to sensitivity $r\Gamma_t$ ("incentives"),
  - promises have to be paid eventually $\longrightarrow$ transversality condition.

Method: Martingale Representation Theorem

- **Definition:** *M is a martingale if* $E\left[\, M_{t+s} \,\middle|\, \mathcal{F}_t \right] = M_t$.
- **Theorem:** *Let Z_t be a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{Q})$ and $\mathcal{F}_t$ the filtration generated by this Brownian motion. If M_t is a martingale with respect to this filtration, then there is an $\mathcal{F}_t$-adapted process Γ such that*

$$M_t = M_0 + \int_0^t \Gamma_s dZ_s, \; 0 \leq t \leq T.$$

Proof of Proposition 1

- Define the expected (average) lifetime utility evaluated conditional on time t information:

$$\begin{aligned} V_t &= E^A\left[r\int_0^\infty e^{-r(s-t)}\left(u(C_s)-h(A_s)\right)ds \,\middle|\, \mathcal{F}_t\right] \\ &= r\int_0^t e^{-rs}\left(u(C_s)-h(A_s)\right)ds + e^{-rt}W_t, \end{aligned}$$

  which is a martingale under $\mathcal{Q}^A$. $\rightarrow$ *Exercise!*

- Applying MRT:

$$V_t = V_0 + r\int_0^t e^{-rs}\Gamma_s \sigma dZ_s^A,$$

  where $Z_t^A = \frac{1}{\sigma}\left(Y_t - \int_0^t A_s ds\right)$ is a Brownian motion under $\mathcal{Q}^A$.

Proof of Proposition 1

- Recall

$$\begin{aligned} V_t &= r\int_0^t e^{-rs}\left(u(C_s)-h(A_s)\right)ds + e^{-rt}W_t \\ &= V_0 + r\int_0^t e^{-rs}\Gamma_s\sigma dZ_s^A. \end{aligned}$$

- Differentiating the two expressions for V_t

$$\begin{aligned} dV_t &= re^{-rt}\left(u(C_t)-h(A_t)\right)dt - re^{-rt}W_t dt + e^{-rt}dW_t \\ &= re^{-rt}\Gamma_t\sigma dZ_t^A, \end{aligned}$$

gives the dynamics of W_t

$$\Leftrightarrow dW_t = r\left(W_t - u(C_t) + h(A_t)\right)dt + r\Gamma_t\underbrace{(dY_t - A_t dt)}_{=\sigma dZ_t^A}.$$

Proof of Proposition 1

- To prove the converse, note that V_t is a martingale when the agent follows A. So:

$$\begin{aligned} W_0 &= V_0 = E\left[V_t\right] \\ &= E\left[r\int_0^t e^{-rs}\left(u(C_s) - h(A_s)\right) ds\right] + E\left[e^{-rt}W_t\right]. \end{aligned}$$

- The result follows by taking the limit as $t \to \infty$

$$W_0 = E\left[r\int_0^\infty e^{-rs}\left(u(C_s) - h(A_s)\right) ds\right].$$

- A similar argument holds for all W_t.

5 Steps to Solve for the Optimal Contract

Step 3:
Necessary and sufficient conditions for the agent's effort level to be optimal (incentive compatibility).

Incentives

- Assume the principal wants to implement effort A_t and recall

$$dW_t = r\left(W_t - u(C_t) + h(A_t)\right)dt + r\Gamma_t\left(dY_t - A_t dt\right).$$

- The agent chooses his true effort $\hat{A}_t$ to maximize

$$E\left[r\left(u(C_t) - h(A_t)\right)dt + dW_t\right],$$

with

$$dW_t = (\text{"}\textit{terms unaffected by deviation}\text{"}) + r\Gamma_t dY_t.$$

- **Proposition 2:** *A contract is incentive compatible if and only if*

$$A_t \in \arg\max_{a \in [0,\overline{A}]} \left(\Gamma_t a - h(a)\right) \;\; \forall t \geq 0.$$

$\rightarrow$ Assuming differentiability Γ_t enforces $A_t > 0$ if

$$\Gamma_t = \gamma(A_t) = h'(A_t).$$

Proof of Proposition 2

- Under contract (C, A), consider an alternative strategy $\hat{A}$ and define

$$\hat{V}_t = r\int_0^t e^{-rs}\left(u(C_s) - h(\hat{A}_s)\right) ds + e^{-rt} W_t(C, A),$$

  the agent's expected payoff from following $\hat{A}$ until time t and A thereafter.

- Differentiating wrt t gives

$$\begin{aligned} d\hat{V}_t &= re^{-rt}\left(u(C_t) - h(\hat{A}_t)\right) dt \underbrace{\begin{array}{c} -re^{-rt}\left(u(C_t) - h(A_t)\right) dt \\ +re^{-rt}\Gamma_t\left(dY_t - A_t dt\right) \end{array}}_{=d(e^{-rt}W_t(C,A))} \\ &= re^{-rt}\left(h(A_t) - h(\hat{A}_t)\right) dt + re^{-rt}\Gamma_t\left(dY_t - A_t dt\right). \end{aligned}$$

Proof of Proposition 2

- If the agent is deviating to $\hat{A}_t$ for an additional moment, then

$$dY_t = \hat{A}_t dt + \sigma dZ_t,$$

and

$$d\hat{V}_t = re^{-rt}\left[\left(h(A_t) - h(\hat{A}_t)\right) + \Gamma_t\left(\hat{A}_t - A_t\right)\right]dt + re^{-rt}\Gamma_t \sigma dZ_t.$$

- Let us now show that if any incremental deviation of this kind hurts the agent, then the whole deviation strategy $\hat{A}$ is worse than A ("one-shot deviation principle").

Proof of Proposition 2

- Claim: A_t is optimal for the agent if and only if:

$$A_t \in \arg\max_{a \in [0, \overline{A}]} \left(\Gamma_t a - h(a)\right) \ \forall t \geq 0. \tag{1}$$

- Drift of $\hat{V}_t$:

$$re^{-rt}\left(\left(\Gamma_t \hat{A}_t - h(\hat{A}_t)\right) - \left(\Gamma_t A_t - h(A_t)\right)\right).$$

- Necessity: If (1) does not hold on a set of positive measure, then choose $\hat{A}_t$ as maximizer in (1) $\rightarrow$ positive drift $\rightarrow$ $\exists t$ such that

$$E^{\hat{A}}\left[\hat{V}_t\right] > \hat{V}_0 = W_0(C, A).$$

- Sufficiency: If (1) does hold, then $\hat{V}_t$ is $\mathcal{Q}^{\hat{A}}$ supermartingale for any $\hat{A}$

$$W_0(C, A) = \hat{V}_0 \geq E^{\hat{A}}\left[\hat{V}_\infty\right] = W_0(C, \hat{A}).$$

5 Steps to Solve for the Optimal Contract

Step 4:
Using a Hamilton Jacobi Bellman (HJB) equation, conjecture an optimal contract.

The Optimal Control Problem

- We now proceed to solve the principal's problem using dynamic programming, with W_t as sole state variable. Intuition:
  - Agent's incentives are unchanged if we replace the continuation contract that follows a given history with a different contract that has the same continuation value.
  - Thus, to maximize the principal's profit after any history, the continuation contract must be optimal given W_t.
- Recall evolution of W_t:

$$dW_t = r\left(W_t - u(C_t) + h(A_t)\right)dt + r\Gamma_t\left(dY_t - A_t dt\right).$$

- The principal
  - controls W_t with C_t and Γ_t (which enforces A_t),
  - must honor promises, i.e. $E\left[e^{-rt}W_t\right] \to 0$ as $t \to \infty$,
  - gets a flow of profits of $r\left(A_t - C_t\right)$.

The Optimal Control Problem

- So, we need to solve the following control problem:

$$F(W_0) = \max\left\{ E\left[r\int_0^{\infty} e^{-r(u-t)}\left(A_u - C_u\right) du\right]\right\},$$

  such that

$$\begin{aligned} dW_t &= r\left(W_t - u(C_t) + h(A_t)\right) dt + r\Gamma_t \left(dY_t - A_t dt\right), \\ W_0 &\; given, \end{aligned}$$

  with maximization over $C_t \geq 0$, $A_t \in \left[0, \overline{A}\right]$ and $\Gamma_t = \gamma(A_t)$ determined from incentive compatibility.

- For a recursive formulation denote by $F(W_t)$ the maximal total profit that the principal can attain from any incentive compatible contract at time t after W_t has been realized.

Deriving the HJB Equation

- Applying the dynamic programing principle, if the principal chooses C_t and A_t optimally, it holds that:

$$F(W_t) = E_t\left[r\int_t^{t+s} e^{-r(u-t)}\left(A_u - C_u\right)du + e^{-rs}F(W_{t+s})\right].$$

- If C_t and A_t are not chosen optimally, then

$$F(W_t) > E_t\left[r\int_t^{t+s} e^{-r(u-t)}\left(A_u - C_u\right)du + e^{-rs}F(W_{t+s})\right].$$

- So, we have

$$F(W_t) = \max_{C,A}\left\{E_t\left[r\int_t^{t+s} e^{-r(u-t)}\left(A_u - C_u\right)du + e^{-rs}F(W_{t+s})\right]\right\}.$$

- We want to derive a differential equation for F.

Method: Itô's Rule

- **Theorem:** *Assume that the process X follows*

$$dX_t = \mu_t dt + \sigma_t dZ_t,$$

with μ and σ adapted processes and let $f(X_t)$ be a twice continuously differentiable function. Then it holds that

$$df(t, X_t) = \left[\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial X} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial X^2}\right] dt + \sigma_t \frac{\partial f}{\partial X} dZ_t,$$

or in integral form

$$f(X_t) = f(X_0) + \int_0^t \left[\frac{\partial f}{\partial t} + \mu_s \frac{\partial f}{\partial X} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial X^2}\right] ds + \int_0^t \sigma_s \frac{\partial f}{\partial X} dZ_s.$$

Deriving the HJB Equation

- Recall, given $W_t = W$ it holds that

$$F(W) \geq E_t \left[r \int_t^{t+s} e^{-r(u-t)} \left(A_u - C_u \right) du + e^{-rs} F(W_{t+s}) \right],$$

with

$$dW_s = r \left(W_s - u(C_s) + h(A_s) \right) ds + r \Gamma_s \sigma dZ_s.$$

- Applying Itô's rule to $e^{-rs} F(W_{t+s})$ we get

$$\begin{aligned} e^{-rs} F(W_{t+s}) &= F(W) + \int_t^{t+s} e^{-r(u-t)} r \Gamma_u \sigma F'(W_u) dZ_u \\ &+ \int_t^{t+s} e^{-r(u-t)} \left[\begin{array}{c} -rF(W_u) + r \left(W_u - u(C_u) + h(A_u) \right) F'(W_u) \\ + \frac{1}{2} r^2 \Gamma_u^2 \sigma^2 F''(W_u) \end{array} \right] du. \end{aligned}$$

- Substituting back in the inequality results in

$$0 \geq E_t \left[r \int_t^{t+s} e^{-r(u-t)} \left[\begin{array}{c} A_u - C_u - F(W_u) + \frac{1}{2} r \Gamma_u^2 \sigma^2 F''(W_u) \\ + \left(W_u - u(C_u) + h(A_u) \right) F'(W_u) \end{array} \right] du \right].$$

Deriving the HJB Equation

- Now divide by s and let $s \to 0$, to arrive at

$$F(W_t) \geq \begin{array}{c} A_t - C_t \\ + \left(W_t - u(C_t) + h(A_t)\right) F'(W_t) + \frac{1}{2} r \Gamma_t^2 \sigma^2 F''(W_t) \end{array}.$$

- This has to hold for all possible $(t, W_t = W)$ and we get the Hamilton Jacobi Bellman equation (HJB)

$$F(W) = \max_{C,A} \left\{ \begin{array}{c} A - C \\ + \left(W - u(C) + h(A)\right) F'(W) + \frac{1}{2} r \Gamma^2 \sigma^2 F''(W) \end{array} \right\},$$

where the maximization is over (admissible) controls $C \geq 0$ and $A \in \left[0, \overline{A}\right]$ subject to incentive compatibility $\Gamma = \gamma(A)$.

The HJB - Intuition

- Assume C_t and A_t are chosen optimally and $W_t = W$ is fixed.
- Since the principal discounts at rate r, his expected flow of value at time t must be $rF(W_t)dt$.
- This has to be equal to
    1. the expected instantaneous flow of output minus payments to the agent $r\left(A_t - C_t\right)dt$,
    2. plus the expected change in the principal's value function $E\left[dF(W_t)\right]$.
- Together we have

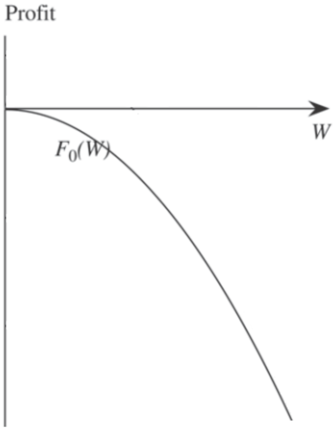
$$rF(W) = \max_{C,A} \left\{ \begin{array}{c} r\left(A - C\right) \\ +r\left(W - u(C) + h(A)\right)F'(W) + \frac{1}{2}r^2\gamma^2(A)\sigma^2 F''(W) \end{array} \right\}.$$

Retirement Value Function

- Always possible to retire the agent:
  - the agent puts zero effort $A_t = 0\ \forall t$,
  - the firm does not produce,
  - the principal offers constant consumption $C_t = C\ \forall t$.
- The principal's retirement profit is

$$F_0(u(C)) = -C,$$

which is decreasing, concave and satisfies $F_0(u(0)) = 0$.



Constructing an Improvement

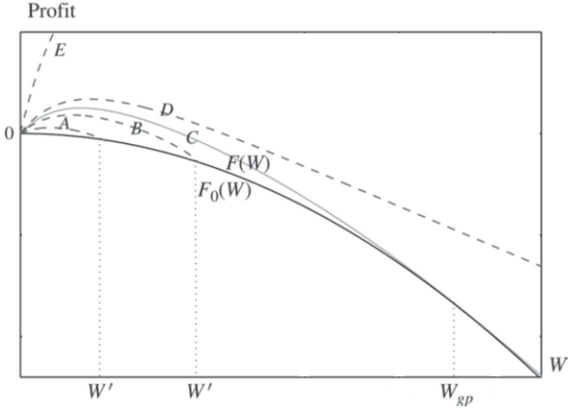
- If W hits zero **have to** retire the agent, as $C \geq 0$.
- If W becomes large, then, due to income effect, it becomes increasingly costly to compensate for effort, hence eventually retire the agent **optimally**.
- Over the improvement interval $A > 0$, and the improvement curve is the solution to the HJB

$$F''(W) = \min_{C,A>0} \frac{F(W) - A + C - (W - u(C) + h(A))\, F'(W)}{r\gamma^2(A)\sigma^2/2},$$

subject to boundary conditions

$$\begin{aligned} F(0) &= 0 && \text{"\textit{value matching}"}, \\ F(W_{gp}) &= F_0(W_{gp}) && \text{"\textit{value matching}"}, \\ F'(W_{gp}) &= F_0'(W_{gp}) && \text{"\textit{smooth pasting}"}. \end{aligned}$$

Constructing an Improvement



- A concave solution $F(W) \geq F_0(W)$ to this boundary value problem exists and is unique.
- The concavity of $F(W)$ is due to the fact that retirement is inefficient.

The Optimal Contract - Summary

- $F(W_0)$ which solves the boundary value problem above is the principal's profit under the optimal contract for $W_0 \in [0, W_{gp}]$.
- The agent's promised wealth under the optimal contract follows

$$\begin{aligned} dW_t &= r\left(W_t - u(c(W_t)) + h(a(W_t))\right) dt \\ &\quad + r\gamma(W_t)\left(dY_t - a(W_t)dt\right) \end{aligned}$$

  until retirement time τ where W_t hits either 0 or W_{gp}.
- For $t < \tau$, $C_t = c(W_t)$ and $A_t = a(W_t)$ are the maximizers in the ODE for $F(W)$.
- After time τ, the agent receives constant consumption $C_t = -F(W_\tau)$ and puts zero effort.

5 Steps to Solve for the Optimal Contract

Step 5:
Verify that the conjectured contract maximizes the principal's profit.

Verification

- So far optimal contract has been conjectured based on a solution of the HJB.
- However, one should note that the HJB takes the form of a necessary condition: "If $F(W)$ is the optimal value function and (C, A) are chosen optimally, then
  - $F(W)$ satisfies the HJB, and
  - The optimal choices of (C, A) realize the maximum in the HJB."
- Further, implicitly made a couple of technical assumptions, in particular on the differentiability of $F(W)$ and the existence of optimal choices of (C, A).
- The verification theorem below will show that the conjectured contract indeed maximizes the principal's profit (sufficiency).

Verification

- Consider the process

$$G_t = r \int_0^t e^{-rs} \left(A_s - C_s \right) ds + e^{-rt} F(W_t).$$

- The drift of G_t is given by

$$re^{-rt} \underbrace{\left[\begin{array}{c} (A_t - C_t) - F(W_t) \\ + \left(W_t - u(C_t) + h(A_t) \right) F'(W_s) + \frac{1}{2} r^2 \Gamma_t^2 \sigma^2 F''(W_s) \end{array} \right]}_{\leq 0 \textit{ from HJB}},$$

which is zero in the conjectured contract and ≤ 0 in any other incentive compatible contract.

- Hence,

$$E \left[r \int_0^\infty e^{-rt} \left(A_t - C_t \right) dt \right] = E \left[G_\infty \right] \leq G_0 = F(W_0),$$

with equality under the optimal contract.

Discussion

Additional Properties of the Optimal Contract:

Initialization, optimal consumption and optimal effort profile.

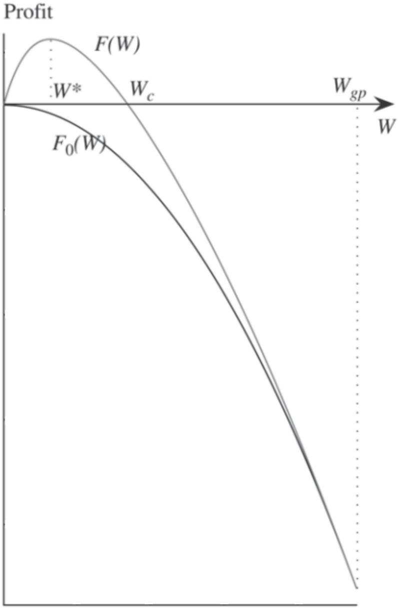
Initialization

- Principal has all bargaining power, $W_0 = W^*$:

$$F'(W^*) = 0.$$

- Agent has all bargaining power, $W_0 = W_c$:

$$F(W_c) = 0.$$



Discussion - Optimal Effort and Consumption

- From the HJB equation, effort maximizes

$$\underbrace{a}_{\text{output}} + \underbrace{h(a)F'(W)}_{\text{cost of compensating for effort}} + \underbrace{\frac{1}{2}r\sigma^2\gamma(a)^2F''(W)}_{\text{cost of providing incentives}}.$$

$\rightarrow$ Effort typically is non-monotonic in W as

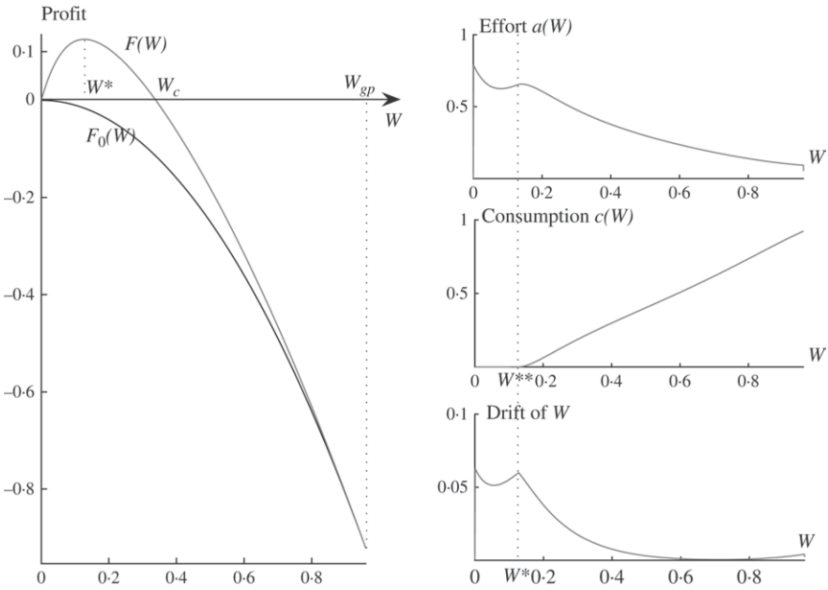
  - $F'(W)$ decreases in W (retirement is inefficient),
  - while $F''(W)$ increases at least for low values of W (exposing agent to risk is costly close to triggering retirement).

- The optimal consumption choice maximizes

$$-c - u(c)F'(W).$$

$\rightarrow$ When $F'(W) \geq -1/u'(0)$, consumption is zero ("probation"). This is the case for $W \in [0, W^{**}]$ (increase drift of W to avoid retirement).
$\rightarrow$ For $W > W^{**}$ consumption is increasing in W.

An Example



Profit
F(W)
W*
W_c
W_gp
W
F_0(W)
0·1
0
−0·2
−0·4
−0·6
−0·8
0
0·2
0·4
0·6
0·8
Effort a(W)
1
0·5
W
0
0·2
0·4
0·6
0·8
Consumption c(W)
1
0·5
W
0
W**
0·2
0·4
0·6
0·8
Drift of W
0·1
0·05
W
0
W*
0·2
0·4
0·6
0·8

Discussion - Optimal Effort and Consumption

- **Proposition 3:** *The drift of W_t points in the direction where $F''(W)$ is increasing, i.e., where it is cheaper to provide incentives.*
- Proof: Differentiating the HJB wrt W using the envelope theorem gives

$$\underbrace{(W - u(C) + h(A))}_{\textit{drift of } W} F''(W) + \frac{1}{2} r\sigma^2\gamma^2(A)F'''(W) = 0. \qquad (2)$$

- Note next that (2) is, from Itô's Lemma, also equal to the drift of $F'(W)$.
  $\rightarrow$ Together with the FOC for (interior consumption)

$$-\frac{1}{u'(c(W))} = F'(W),$$

  this implies that $1/u'(C)$ is a martingale (**"Inverse Euler Equation"**).
- Reflects the fact that agent cannot save: $u'(C)$ is a submartingale.
  $\rightarrow$ So if the agent could save he would want to do so as his marginal utility increases in expectation.

Contractual Environments

How do Contractual Environments Affect Agent's Career?

Contractual Environments

- Different Contractual environments:

  A.) The agent can quit and pursue an outside option,
  B.) the principal can replace the agent,
  C.) the principal can promote the agent.

- Properties of agent's career:

  1.) Wages (back-loaded vs. front-loaded),
  2.) short-term incentives (piece rates, bonuses) vs. long-term incentives (permanent wage increases, terminations),
  3.) the agent's effort in equilibrium.

Solve the Model under Different Environments

- Principal's generalized problem: Maximize profit until $t = \tau$ when the agent quits, retires, is replaced, or promoted

$$E\left[r\int_0^{\tau} e^{-rt}\left(A_t - C_t\right)dt + e^{-r\tau}\tilde{F}_0\left(W_\tau\right)\right],$$

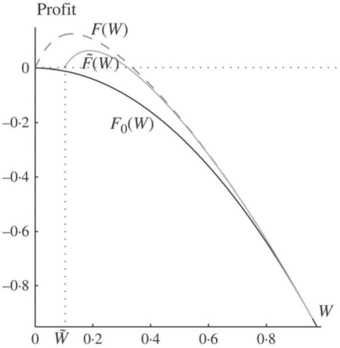
  subject to incentive compatibility constraint and the agent's participation constraint for all $t \leq \tau$,

$$W_t \geq \tilde{W} \geq 0.$$

- The principal's profit function $\tilde{F}\left(W\right)$ has to satisfy the same HJB as before, but the respective environment determines the boundary conditions:

$$\tilde{F}\left(W_\tau\right) = \tilde{F}_0\left(W_\tau\right).$$

A.) Profit Function with Outside Option



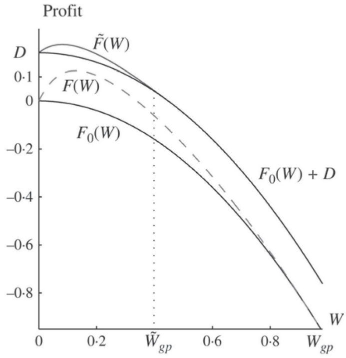
- Lower retirement point is **higher** than w/o outside option:

$$\tilde{W} > 0.$$

- Principal's profit is **lower** than w/o outside option:

$$\tilde{F}(W) < F(W).$$

B.) Profit Function with Replacement



- Retirement profit **higher** than w/o replacement:

$$\tilde{F}_0(W) = F_0(W) + D.$$

- Principal's profit is **higher** than w/o replacement:

$$\tilde{F}(W) > F(W).$$

- Less costly to retire the agent → upper retirement point **lower** than w/o replacement:

$$\tilde{W}_{gp} < W_{gp}.$$

C.) Promotion of the Agent

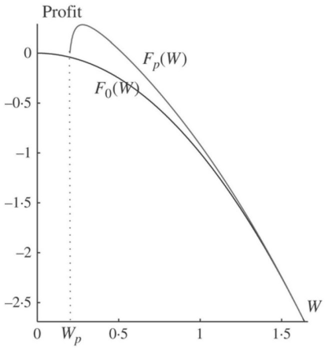
- Promoting the agent to a new position
  - incurs the principal training cost K,
  - increases the agent's productivity by a factor of $\theta > 1$,
  - Increases the agent's outside option to $W_p > 0$.
- With a promoted agent, the principal's profit function solves

$$F_p^{''}(W) = \min_{C,A>0} \frac{F_p(W) - \theta A + C - (W - u(C) + h(A))\, F_p'(W)}{r\gamma^2(A)\sigma^2 / \left(2\theta^2\right)},$$

with boundary conditions

$$\begin{aligned} F_p(\tilde{W}_p) &= 0, \\ F_p(W_{gp}) &= F_0(W_{gp}), \\ F_p'(W_{gp}) &= F_0'(W_{gp}). \end{aligned}$$

C.) Profit Function after Promotion

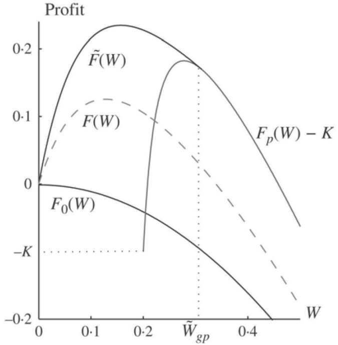


- Lower retirement point is **higher** than w/o promotion (agent now has an outside option):

$$W_p > 0.$$

- Upper retirement point is also **higher** than w/o promotion because a trained agent is more productive.

C.) Profit Function before Promotion



- Principal must decide whether to promote or to retire the agent:

$$\tilde{F}_0(W) = \max\left(F_0(W), F_p(W) - K\right).$$

- Here: Agent is promoted at $\tilde{W}_{gp}$ where:

$$\begin{aligned} \tilde{F}\left(\tilde{W}_{gp}\right) &= F_p\left(\tilde{W}_{gp}\right) - K, \\ \tilde{F}'\left(\tilde{W}_{gp}\right) &= F_p'\left(\tilde{W}_{gp}\right). \end{aligned}$$

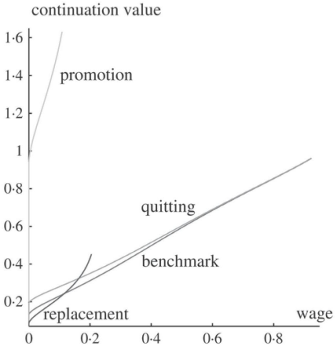
- Principal's profit is **higher** than w/o promotion:

$$\tilde{F}(W) > F(W).$$

1.) Front-Loaded vs. Back-Loaded Compensation

- A fully dynamic setting allows us to study when wages should be more front-loaded and when they should be more back-loaded.
- E.g. Lazear (1979) shows that:
  - The employers can strengthen an employment relationship by offering a rising wage pattern.
  - By postponing pay to a later point in the agent's career, he can be induced to exert more effort at the same costs for the principal.
- In the present setting:
  - The Optimal contract trades off this benefit against costs from
    - income effect,
    - earlier retirement, and
    - distortion of agent's consumption.

1.) Front-Loaded vs. Back-Loaded Compensation



- Measure for how back-loaded the agent's compensation is:
- **wage** captures **short-term** compensation.
- **continuation value** captures **long-term** compensation.

$\longrightarrow$ compare environments by looking at continuation value for a given wage.

2.) Short-Term Incentives vs. Long-Term Incentives

- Long-term and short-term incentives have been studied individually.
  - Short-term incentives:
    - Holmström and Milgrom (1987) "especially well suited for representing compensation paid over short period" (from HM 1991).
    - Lazear (2000): productivity in Safelite Glass Corporation increased by 44 % when piece rates were introduced.
  - Long-term incentives:
    - Lazear and Rosen (1981): incentives can be created by promotions.
- Optimal mix of short-term and long-term incentives has not been studied.

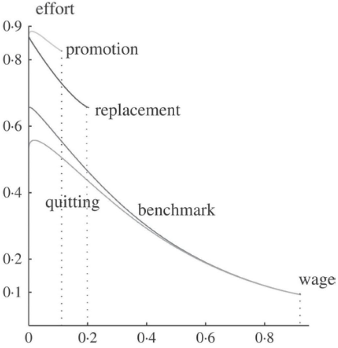
2.) Short-Term Incentives vs. Long-Term Incentives

- Incentives are provided by tying the agent's compensation to the project's risky outcome.
  - Volatility of current **consumption** captures **short-term** incentives.
  - Volatility of **continuation value** captures **long-term** incentives.

$\longrightarrow$ Use the relative volatility of the agent's compensation as a measure for the dynamics of incentive provision.

  - Agent has **outside option** $\Rightarrow$ **less** long-term incentives.
  - Principal can **replace** the agent $\Rightarrow$ **more** long-term incentives.
  - Principal can **promote** the agent $\Rightarrow$ **more** long-term incentives.

3.) Equilibrium Effort Profile



- Higher effort when the optimal contract relies more on long-term incentives.

Sannikov (2008) Conclusions

- Clean and elegant method to study dynamic incentive problems.
- Linear over short periods as in Holmström and Milgrom (1987) but nonlinear in the long run.
- How does contractual environment affect dynamics.
- Next: Look at a dynamic model of financial contracting with risk-neutrality (DeMarzo and Sannikov 2006).