

# **Equilibrium Theory in Continuous Time**

Lecture Notes

Tomas Björk

Stockholm School of Economics

[tomas.bjork@hhs.se](mailto:tomas.bjork@hhs.se)

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Part I

# Portfolio Optimization



# Chapter 1

## Stochastic Optimal Control

### 1.1 An Example

Let us consider an economic agent over a fixed time interval  $[0, T]$ . At time  $t = 0$  the agent is endowed with initial wealth  $x_0$  and his/her problem is how to allocate investments and consumption over the given time horizon. We assume that the agent's investment opportunities are the following.

- The agent can invest money in the bank at the deterministic short rate of interest  $r$ , i.e. he/she has access to the risk free asset  $B$  with

$$dB = rBdt. \tag{1.1}$$

- The agent can invest in a risky asset with price process  $S_t$ , where we assume that the  $S$ -dynamics are given by a standard Black–Scholes model

$$dS = \alpha Sdt + \sigma SdW. \tag{1.2}$$

We denote the agent's relative portfolio weights at time  $t$  by  $u_t^0$  (for the riskless asset), and  $u_t^1$  (for the risky asset) respectively. His/her consumption rate at time  $t$  is denoted by  $c_t$ .

We restrict the consumer's investment–consumption strategies to be self-financing, and as usual we assume that we live in a world where continuous trading and unlimited short selling is possible. If we denote the wealth of the consumer at time  $t$  by  $X_t$ , it now follows from general portfolio theory that (after a slight rearrangement of terms) the  $X$ -dynamics are given by

$$dX_t = X_t [u_t^0 r + u_t^1 \alpha] dt - c_t dt + u_t^1 \sigma X_t dW_t. \tag{1.3}$$

The object of the agent is to choose a portfolio–consumption strategy in such a way as to maximize his/her total utility over  $[0, T]$ , and we assume that this utility is given by

$$E \left[ \int_0^T F(t, c_t) dt + \Phi(X_T) \right], \tag{1.4}$$

where  $F$  is the instantaneous utility function for consumption, whereas  $\Phi$  is a “legacy” function which measures the utility of having some money left at the end of the period.

A natural constraint on consumption is the condition

$$c_t \geq 0, \quad \forall t \geq 0, \quad (1.5)$$

and we also have of course the constraint

$$u_t^0 + u_t^1 = 1, \quad \forall t \geq 0. \quad (1.6)$$

Depending upon the actual situation we may be forced to impose other constraints (it may, say, be natural to demand that the consumer’s wealth never becomes negative), but we will not do this at the moment.

We may now formally state the consumer’s utility maximization problem as follows.

$$\max_{u^0, u^1, c} E \left[ \int_0^T F(t, c_t) dt + \Phi(X_T) \right] \quad (1.7)$$

$$dX_t = X_t [u_t^0 r + u_t^1 \alpha] dt - c_t dt + u_t^1 \sigma X_t dW_t, \quad (1.8)$$

$$X_0 = x_0, \quad (1.9)$$

$$c_t \geq 0, \quad \forall t \geq 0, \quad (1.10)$$

$$u_t^0 + u_t^1 = 1, \quad \forall t \geq 0. \quad (1.11)$$

A problem of this kind is known as a **stochastic optimal control problem**. In this context the process  $X$  is called the **state process** (or state variable), the processes  $u^0$ ,  $u^1$ ,  $c$  are called **control processes**, and we have a number of **control constraints**. In the next sections we will study a fairly general class of stochastic optimal control problems. The method used is that of **dynamic programming**, and at the end of the chapter we will solve a version of the problem above.

## 1.2 The Formal Problem

We now go on to study a fairly general class of optimal control problems. To this end, let  $\mu(t, x, u)$  and  $\sigma(t, x, u)$  be given functions of the form

$$\begin{aligned} \mu &: R_+ \times R^n \times R^k \rightarrow R^n, \\ \sigma &: R_+ \times R^n \times R^k \rightarrow R^{n \times d}. \end{aligned}$$

For a given point  $x_0 \in R^n$  we will consider the following **controlled** stochastic differential equation.

$$dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t, \quad (1.12)$$

$$X_0 = x_0. \quad (1.13)$$

We view the  $n$ -dimensional process  $X$  as a **state process**, which we are trying to “control” (or “steer”). We can (partly) control the state process  $X$  by choosing the  $k$ -dimensional **control process**  $u$  in a suitable way.  $W$  is a  $d$ -dimensional Wiener process, and we must now try to give a precise mathematical meaning to the formal expressions (1.12)–(1.13).

**Remark 1.2.1** *In this chapter, where we will work under a fixed measure, all Wiener processes are denoted by the letter  $W$ .*

Our first modelling problem concerns the class of admissible control processes. In most concrete cases it is natural to require that the control process  $u$  is adapted to the  $X$  process. In other words, at time  $t$  the value  $u_t$  of the control process is only allowed to “depend” on past observed values of the state process  $X$ . One natural way to obtain an adapted control process is by choosing a deterministic function  $g(t, x)$

$$g : R_+ \times R^n \rightarrow R^k,$$

and then defining the control process  $u$  by

$$u_t = g(t, X_t).$$

Such a function  $g$  is called a **feedback control law**, and in the sequel we will restrict ourselves to consider only feedback control laws. For mnemo-technical purposes we will often denote control laws by  $\mathbf{u}(t, x)$ , rather than  $g(t, x)$ , and write  $u_t = \mathbf{u}(t, X_t)$ . We use boldface in order to indicate that  $\mathbf{u}$  is a **function**. In contrast to this we use the notation  $u$  (italics) to denote the **value** of a control at a certain time. Thus  $\mathbf{u}$  denotes a mapping, whereas  $u$  denotes a point in  $R^k$ .

Suppose now that we have chosen a fixed control law  $\mathbf{u}(t, x)$ . Then we can insert  $\mathbf{u}$  into (1.12) to obtain the standard SDE

$$dX_t = \mu(t, X_t, \mathbf{u}(t, X_t)) dt + \sigma(t, X_t, \mathbf{u}(t, X_t)) dW_t. \quad (1.14)$$

In most concrete cases we also have to satisfy some **control constraints**, and we model this by taking as given a fixed subset  $U \subseteq R^k$  and requiring that  $u_t \in U$  for each  $t$ . We can now define the class of **admissible control laws**.

**Definition 1.2.1** *A control law  $\mathbf{u}$  is called **admissible** if*

- $\mathbf{u}(t, x) \in U$  for all  $t \in R_+$  and all  $x \in R^n$ .
- For any given initial point  $(t, x)$  the SDE

$$\begin{aligned} dX_s &= \mu(s, X_s, \mathbf{u}(s, X_s)) ds + \sigma(s, X_s, \mathbf{u}(s, X_s)) dW_s, \\ X_t &= x \end{aligned}$$

*has a unique solution.*

*The class of admissible control laws is denoted by  $\mathcal{U}$ .*

For a given control law  $\mathbf{u}$ , the solution process  $X$  will of course depend on the initial value  $x$ , as well as on the chosen control law  $\mathbf{u}$ . To be precise we should therefore denote the process  $X$  by  $X^{x,\mathbf{u}}$ , but sometimes we will suppress  $x$  or  $\mathbf{u}$ . We note that eqn (1.14) looks rather messy, and since we will also have to deal with the Itô formula in connection with (1.14) we need some more streamlined notation.

**Definition 1.2.2** Consider eqn (1.14), and let  $'$  denote matrix transpose.

- For any fixed vector  $u \in R^k$ , the functions  $\mu^u$ ,  $\sigma^u$  and  $C^u$  are defined by

$$\begin{aligned}\mu^u(t, x) &= \mu(t, x, u), \\ \sigma^u(t, x) &= \sigma(t, x, u), \\ C^u(t, x) &= \sigma(t, x, u)\sigma(t, x, u)'.\end{aligned}$$

- For any control law  $\mathbf{u}$ , the functions  $\mu^{\mathbf{u}}$ ,  $\sigma^{\mathbf{u}}$ ,  $C^{\mathbf{u}}(t, x)$  and  $F^{\mathbf{u}}(t, x)$  are defined by

$$\begin{aligned}\mu^{\mathbf{u}}(t, x) &= \mu(t, x, \mathbf{u}(t, x)), \\ \sigma^{\mathbf{u}}(t, x) &= \sigma(t, x, \mathbf{u}(t, x)), \\ C^{\mathbf{u}}(t, x) &= \sigma(t, x, \mathbf{u}(t, x))\sigma(t, x, \mathbf{u}(t, x))', \\ F^{\mathbf{u}}(t, x) &= F(t, x, \mathbf{u}(t, x)).\end{aligned}$$

- For any fixed vector  $u \in R^k$ , the partial differential operator  $\mathcal{A}^u$  is defined by

$$\mathcal{A}^u = \sum_{i=1}^n \mu_i^u(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^u(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law  $\mathbf{u}$ , the partial differential operator  $\mathcal{A}^{\mathbf{u}}$  is defined by

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^n \mu_i^{\mathbf{u}}(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^{\mathbf{u}}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Given a control law  $\mathbf{u}$  we will sometimes write eqn (1.14) in a convenient shorthand notation as

$$dX_t^{\mathbf{u}} = \mu^{\mathbf{u}} dt + \sigma^{\mathbf{u}} dW_t. \quad (1.15)$$

For a given control law  $\mathbf{u}$  with a corresponding controlled process  $X^{\mathbf{u}}$  we will also often use the shorthand notation  $\mathbf{u}_t$  instead of the clumsier expression  $\mathbf{u}(t, X_t^{\mathbf{u}})$ .

The reader should be aware of the fact that the existence assumption in the definition above is not at all an innocent one. In many cases it is natural to consider control laws which are “rapidly varying”, i.e. feedback laws  $\mathbf{u}(t, x)$  which are very irregular as functions of the state variable  $x$ . Inserting such an irregular control law into the state dynamics will easily give us a very irregular

drift function  $\mu(t, x, \mathbf{u}(t, x))$  (as a function of  $x$ ), and we may find ourselves outside the nice standard Lipschitz situation, thus leaving us with a highly nontrivial existence problem. The reader is referred to the literature for details.

We now go on to the objective function of the control problem, and therefore we consider as given a pair of functions

$$\begin{aligned} F &: R_+ \times R^n \times R^k \rightarrow R, \\ \Phi &: R^n \rightarrow R. \end{aligned}$$

Now we define the **value function** of our problem as the function

$$\mathcal{J}_0 : \mathcal{U} \rightarrow R,$$

defined by

$$\mathcal{J}_0(\mathbf{u}) = E \left[ \int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dt + \Phi(X_T^{\mathbf{u}}) \right],$$

where  $X^{\mathbf{u}}$  is the solution to (1.14) with the given initial condition  $X_0 = x_0$ .

Our formal problem can thus be written as that of maximizing  $\mathcal{J}_0(\mathbf{u})$  over all  $\mathbf{u} \in \mathcal{U}$ , and we define the **optimal value**  $\hat{\mathcal{J}}_0$  by

$$\hat{\mathcal{J}}_0 = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}_0(\mathbf{u}).$$

If there exists an admissible control law  $\hat{\mathbf{u}}$  with the property that

$$\mathcal{J}_0(\hat{\mathbf{u}}) = \hat{\mathcal{J}}_0,$$

then we say that  $\hat{\mathbf{u}}$  is an **optimal control law** for the given problem. Note that, as for any optimization problem, the optimal law may not exist. For a given concrete control problem our main objective is of course to find the optimal control law (if it exists), or at least to learn something about the qualitative behavior of the optimal law.

## 1.3 The Hamilton–Jacobi–Bellman Equation

Given an optimal control problem we have two natural questions to answer:

- (a) Does there exist an optimal control law?
- (b) Given that an optimal control exists, how do we find it?

In this text we will mainly be concerned with problem (b) above, and the methodology used will be that of **dynamic programming**. The main idea is to embed our original problem into a much larger class of problems, and then to tie all these problems together with a partial differential equation (PDE) known as the Hamilton–Jacobi–Bellman equation. The control problem is then shown to be equivalent to the problem of finding a solution to the HJB equation.

We will now describe the embedding procedure, and for that purpose we choose a fixed point  $t$  in time, with  $0 \leq t \leq T$ . We also choose a fixed point  $x$  in the state space, i.e.  $x \in R^n$ . For this fixed pair  $(t, x)$  we now define the following control problem.

**Definition 1.3.1** *The control problem  $\mathcal{P}(t, x)$  is defined as the problem to maximize*

$$E_{t,x} \left[ \int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi(X_T^{\mathbf{u}}) \right], \quad (1.16)$$

*given the dynamics*

$$dX_s^{\mathbf{u}} = \mu(s, X_s^{\mathbf{u}}, \mathbf{u}(s, X_s^{\mathbf{u}})) ds + \sigma(s, X_s^{\mathbf{u}}, \mathbf{u}(s, X_s^{\mathbf{u}})) dW_s, \quad (1.17)$$

$$X_t = x, \quad (1.18)$$

*and the constraints*

$$\mathbf{u}(s, y) \in U, \quad \forall (s, y) \in [t, T] \times R^n. \quad (1.19)$$

Observe that we use the notation  $s$  and  $y$  above because the letters  $t$  and  $x$  are already used to denote the fixed chosen point  $(t, x)$ .

We note that in terms of the definition above, our original problem is the problem  $\mathcal{P}(0, x_0)$ . A somewhat drastic interpretation of the problem  $\mathcal{P}(t, x)$  is that you have fallen asleep at time zero. Suddenly you wake up, noticing that the time now is  $t$  and that your state process while you were asleep has moved to the point  $x$ . You now try to do as well as possible under the circumstances, so you want to maximize your utility over the remaining time, given the fact that you start at time  $t$  in the state  $x$ .

We now define the **value function** and the **optimal value function**.

**Definition 1.3.2**

- *The value function*

$$\mathcal{J} : R_+ \times R^n \times \mathcal{U} \rightarrow R$$

*is defined by*

$$\mathcal{J}(t, x, \mathbf{u}) = E \left[ \int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi(X_T^{\mathbf{u}}) \right]$$

*given the dynamics (1.17)–(1.18).*

- *The optimal value function*

$$V : R_+ \times R^n \rightarrow R$$

*is defined by*

$$V(t, x) = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t, x, \mathbf{u}).$$



Thus  $\mathcal{J}(t, x, \mathbf{u})$  is the expected utility of using the control law  $\mathbf{u}$  over the time interval  $[t, T]$ , given the fact that you start in state  $x$  at time  $t$ . The optimal value function gives you the optimal expected utility over  $[t, T]$  under the same initial conditions.

The main object of interest for us is the optimal value function, and we now go on to derive a PDE for  $V$ . It should be noted that this derivation is largely heuristic. We make some rather strong regularity assumptions, and we disregard a number of technical problems. We will comment on these problems later, but to see exactly which problems we are ignoring we now make some basic assumptions.

**Assumption 1.3.1** *We assume the following.*

1. *There exists an optimal control law  $\hat{\mathbf{u}}$ .*
2. *The optimal value function  $V$  is regular in the sense that  $V \in C^{1,2}$ .*
3. *A number of limiting procedures in the following arguments can be justified.*

We now go on to derive the PDE, and to this end we fix  $(t, x) \in (0, T) \times R^n$ . Furthermore we choose a real number  $h$  (interpreted as a “small” time increment) such that  $t + h < T$ . We choose a fixed but arbitrary control law  $\mathbf{u}$ , and define the control law  $\mathbf{u}^*$  by

$$\mathbf{u}^*(s, y) = \begin{cases} \mathbf{u}(s, y), & (s, y) \in [t, t + h] \times R^n \\ \hat{\mathbf{u}}(s, y), & (s, y) \in (t + h, T) \times R^n. \end{cases}$$

In other words, if we use  $\mathbf{u}^*$  then we use the arbitrary control  $\mathbf{u}$  during the time interval  $[t, t + h]$ , and then we switch to the optimal control law during the rest of the time period.

The whole idea of dynamic programming actually boils down to the following procedure.

- First, given the point  $(t, x)$  as above, we consider the following two strategies over the time interval  $[t, T]$ :

**Strategy I.** Use the optimal law  $\hat{\mathbf{u}}$ .

**Strategy II.** Use the control law  $\mathbf{u}^*$  defined above.

- We then compute the expected utilities obtained by the respective strategies.
- Finally, using the obvious fact that Strategy I by definition has to be at least as good as Strategy II, and letting  $h$  tend to zero, we obtain our fundamental PDE.

We now carry out this program.

**Expected utility for strategy I:** This is trivial, since by definition the utility is the optimal one given by  $\mathcal{J}(t, x, \hat{\mathbf{u}}) = V(t, x)$ .

**Expected utility for strategy II:** We divide the time interval  $[t, T]$  into two parts, the intervals  $[t, t+h]$  and  $(t+h, T]$  respectively.

- The expected utility, using Strategy II, for the interval  $[t, t+h]$  is given by

$$E_{t,x} \left[ \int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds \right].$$

- In the interval  $[t+h, T]$  we observe that at time  $t+h$  we will be in the (stochastic) state  $X_{t+h}^{\mathbf{u}}$ . Since, by definition, we will use the optimal strategy during the entire interval  $[t+h, T]$  we see that the remaining expected utility at time  $t+h$  is given by  $V(t+h, X_{t+h}^{\mathbf{u}})$ . Thus the expected utility over the interval  $[t+h, T]$ , conditional on the fact that at time  $t$  we are in state  $x$ , is given by

$$E_{t,x} [V(t+h, X_{t+h}^{\mathbf{u}})].$$

Thus the total expected utility for Strategy II is

$$E_{t,x} \left[ \int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + V(t+h, X_{t+h}^{\mathbf{u}}) \right].$$

**Comparing the strategies:** We now go on to compare the two strategies, and since by definition Strategy I is the optimal one, we must have the inequality

$$V(t, x) \geq E_{t,x} \left[ \int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + V(t+h, X_{t+h}^{\mathbf{u}}) \right]. \quad (1.20)$$

We also note that the inequality sign is due to the fact that the arbitrarily chosen control law  $\mathbf{u}$  which we use on the interval  $[t, t+h]$  need not be the optimal one. In particular we have the following obvious fact.

**Remark 1.3.1** *We have equality in (1.20) if and only if the control law  $\mathbf{u}$  is an optimal law  $\hat{\mathbf{u}}$ . (Note that the optimal law does not have to be unique.)*

Since, by assumption,  $V$  is smooth we now use the Itô formula to obtain (with obvious notation)

$$\begin{aligned} V(t+h, X_{t+h}^{\mathbf{u}}) &= V(t, x) + \int_t^{t+h} \left\{ \frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}} V(s, X_s^{\mathbf{u}}) \right\} ds \\ &\quad + \int_t^{t+h} \nabla_x V(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s. \end{aligned} \quad (1.21)$$

If we apply the expectation operator  $E_{t,x}$  to this equation, and assume enough integrability, then the stochastic integral will vanish. We can then

insert the resulting equation into the inequality (1.20). The term  $V(t, x)$  will cancel, leaving us with the inequality

$$E_{t,x} \left[ \int_t^{t+h} \left[ F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) + \frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}}V(s, X_s^{\mathbf{u}}) \right] ds \right] \leq 0. \quad (1.22)$$

**Going to the limit:** Now we divide by  $h$ , move  $h$  within the expectation and let  $h$  tend to zero. Assuming enough regularity to allow us to take the limit within the expectation, using the fundamental theorem of integral calculus, and recalling that  $X_t = x$ , we get

$$F(t, x, u) + \frac{\partial V}{\partial t}(t, x) + \mathcal{A}^u V(t, x) \leq 0, \quad (1.23)$$

where  $u$  denotes the value of the law  $\mathbf{u}$  evaluated at  $(t, x)$ , i.e.  $u = \mathbf{u}(t, x)$ . Since the control law  $\mathbf{u}$  was arbitrary, this inequality will hold for all choices of  $u \in U$ , and we will have equality if and only if  $u = \hat{\mathbf{u}}(t, x)$ . We thus have the following equation

$$\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0.$$

During the discussion the point  $(t, x)$  was fixed, but since it was chosen as an arbitrary point we see that the equation holds in fact for all  $(t, x) \in (0, T) \times R^n$ . Thus we have a (nonstandard type of) PDE, and we obviously need some boundary conditions. One such condition is easily obtained, since we obviously (why?) have  $V(T, x) = \Phi(x)$  for all  $x \in R^n$ . We have now arrived at our goal, namely the Hamilton–Jacobi–Bellman equation, (often referred to as the HJB equation.)

**Theorem 1.3.1 (Hamilton–Jacobi–Bellman equation)** *Under Assumption 1.3.1, the following hold.*

1.  $V$  satisfies the Hamilton–Jacobi–Bellman equation

$$\begin{cases} \frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0, & \forall (t, x) \in (0, T) \times R^n \\ V(T, x) = \Phi(x), & \forall x \in R^n. \end{cases}$$

2. For each  $(t, x) \in [0, T] \times R^n$  the supremum in the HJB equation above is attained by  $u = \hat{\mathbf{u}}(t, x)$ .

**Remark 1.3.2** *By going through the arguments above, it is easily seen that we may allow the constraint set  $U$  to be time- and state-dependent. If we thus have control constraints of the form*

$$u(t, x) \in U(t, x), \quad \forall t, x$$

*then the HJB equation still holds with the obvious modification of the supremum part.*

It is important to note that this theorem has the form of a **necessary** condition. It says that **if**  $V$  is the optimal value function, and **if**  $\hat{\mathbf{u}}$  is the optimal control, **then**  $V$  satisfies the HJB equation, and  $\hat{\mathbf{u}}(t, x)$  realizes the supremum in the equation. We also note that Assumption 1.3.1 is an *ad hoc* assumption. One would prefer to have conditions in terms of the initial data  $\mu$ ,  $\sigma$ ,  $F$  and  $\Phi$  which would guarantee that Assumption 1.3.1 is satisfied. This can in fact be done, but at a fairly high price in terms of technical complexity. The reader is referred to the specialist literature.

A gratifying, and perhaps surprising, fact is that the HJB equation also acts as a **sufficient** condition for the optimal control problem. This result is known as the **verification theorem** for dynamic programming, and we will use it repeatedly below. Note that, as opposed to the necessary conditions above, the verification theorem is very easy to prove rigorously.

**Theorem 1.3.2 (Verification theorem)** *Suppose that we have two functions  $H(t, x)$  and  $g(t, x)$ , such that*

- $H$  is sufficiently integrable (see Remark 1.3.4 below), and solves the HJB equation

$$\begin{cases} \frac{\partial H}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\} = 0, & \forall (t, x) \in (0, T) \times R^n \\ H(T, x) = \Phi(x), & \forall x \in R^n. \end{cases}$$

- The function  $g$  is an admissible control law.
- For each fixed  $(t, x)$ , the supremum in the expression

$$\sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\}$$

is attained by the choice  $u = g(t, x)$ .

Then the following hold.

1. The optimal value function  $V$  to the control problem is given by

$$V(t, x) = H(t, x).$$

2. There exists an optimal control law  $\hat{\mathbf{u}}$ , and in fact  $\hat{\mathbf{u}}(t, x) = g(t, x)$ .

**Remark 1.3.3** *Note that we have used the letter  $H$  (instead of  $V$ ) in the HJB equation above. This is because the letter  $V$  by definition denotes the optimal value function.*

**Proof.** Assume that  $H$  and  $g$  are given as above. Now choose an arbitrary control law  $\mathbf{u} \in \mathcal{U}$ , and fix a point  $(t, x)$ . We define the process  $X^{\mathbf{u}}$  on the time interval  $[t, T]$  as the solution to the equation

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu^{\mathbf{u}}(s, X_s^{\mathbf{u}}) ds + \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) dW_s, \\ X_t &= x. \end{aligned}$$

Inserting the process  $X^{\mathbf{u}}$  into the function  $H$  and using the Itô formula we obtain

$$\begin{aligned} H(T, X_T^{\mathbf{u}}) &= H(t, x) + \int_t^T \left\{ \frac{\partial H}{\partial t}(s, X_s^{\mathbf{u}}) + (\mathcal{A}^{\mathbf{u}}H)(s, X_s^{\mathbf{u}}) \right\} ds \\ &\quad + \int_t^T \nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) dW_s. \end{aligned}$$

Since  $H$  solves the HJB equation we see that

$$\frac{\partial H}{\partial t}(t, x) + F(t, x, u) + \mathcal{A}^u H(t, x) \leq 0$$

for all  $u \in U$ , and thus we have, for each  $s$  and  $P$ -a.s, the inequality

$$\frac{\partial H}{\partial t}(s, X_s^{\mathbf{u}}) + (\mathcal{A}^{\mathbf{u}}H)(s, X_s^{\mathbf{u}}) \leq -F^{\mathbf{u}}(s, X_s^{\mathbf{u}}).$$

From the boundary condition for the HJB equation we also have  $H(T, X_T^{\mathbf{u}}) = \Phi(X_T^{\mathbf{u}})$ , so we obtain the inequality

$$H(t, x) \geq \int_t^T F^{\mathbf{u}}(s, X_s^{\mathbf{u}}) ds + \Phi(X_T^{\mathbf{u}}) - \int_t^T \nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s.$$

Taking expectations, and assuming enough integrability, we make the stochastic integral vanish, leaving us with the inequality

$$H(t, x) \geq E_{t,x} \left[ \int_t^T F^{\mathbf{u}}(s, X_s^{\mathbf{u}}) ds + \Phi(X_T^{\mathbf{u}}) \right] = \mathcal{J}(t, x, \mathbf{u}).$$

Since the control law  $\mathbf{u}$  was arbitrarily chosen this gives us

$$H(t, x) \geq \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t, x, \mathbf{u}) = V(t, x). \quad (1.24)$$

To obtain the reverse inequality we choose the specific control law  $\mathbf{u}(t, x) = \mathbf{g}(t, x)$ . Going through the same calculations as above, and using the fact that by assumption we have

$$\frac{\partial H}{\partial t}(t, x) + F^{\mathbf{g}}(t, x) + \mathcal{A}^{\mathbf{g}}H(t, x) = 0,$$

we obtain the equality

$$H(t, x) = E_{t,x} \left[ \int_t^T F^{\mathbf{g}}(s, X_s^{\mathbf{g}}) ds + \Phi(X_T^{\mathbf{g}}) \right] = \mathcal{J}(t, x, \mathbf{g}). \quad (1.25)$$

On the other hand we have the trivial inequality

$$V(t, x) \geq \mathcal{J}(t, x, \mathbf{g}), \quad (1.26)$$

so, using (1.24)–(1.26), we obtain

$$H(t, x) \geq V(t, x) \geq \mathcal{J}(t, x, \mathbf{g}) = H(t, x).$$

This shows that in fact

$$H(t, x) = V(t, x) = \mathcal{J}(t, x, \mathbf{g}),$$

which proves that  $H = V$ , and that  $\mathbf{g}_s$  is the optimal control law. ■

**Remark 1.3.4** *The assumption that  $H$  is “sufficiently integrable” in the theorem above is made in order for the stochastic integral in the proof to have expected value zero. This will be the case if, for example,  $H$  satisfies the condition*

$$\nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) \in \mathcal{L}^2,$$

for all admissible control laws.

**Remark 1.3.5** *Sometimes, instead of a maximization problem, we consider a minimization problem. Of course we now make the obvious definitions for the value function and the optimal value function. It is then easily seen that all the results above still hold if the expression*

$$\sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\}$$

in the HJB equation is replaced by the expression

$$\inf_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\}.$$

**Remark 1.3.6** *In the Verification Theorem we may allow the control constraint set  $U$  to be state and time dependent, i.e. of the form  $U(t, x)$ .*

## 1.4 Handling the HJB Equation

In this section we will describe the actual handling of the HJB equation, and in the next section we will study a classical example—the linear quadratic regulator. We thus consider our standard optimal control problem with the corresponding HJB equation:

$$\begin{cases} \frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0, \\ V(T, x) = \Phi(x). \end{cases} \quad (1.27)$$

Schematically we now proceed as follows.

1. Consider the HJB equation as a PDE for an unknown function  $V$ .

2. Fix an arbitrary point  $(t, x) \in [0, T] \times R^n$  and solve, for this fixed choice of  $(t, x)$ , the static optimization problem

$$\max_{u \in U} [F(t, x, u) + \mathcal{A}^u V(t, x)].$$

Note that in this problem  $u$  is the only variable, whereas  $t$  and  $x$  are considered to be fixed parameters. The functions  $F$ ,  $\mu$ ,  $\sigma$  and  $V$  are considered as given.

3. The optimal choice of  $u$ , denoted by  $\hat{u}$ , will of course depend on our choice of  $t$  and  $x$ , but it will also depend on the function  $V$  and its various partial derivatives (which are hiding under the sign  $\mathcal{A}^u V$ ). To highlight these dependencies we write  $\hat{u}$  as

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, x; V). \quad (1.28)$$

4. The function  $\hat{u}(t, x; V)$  is our candidate for the optimal control law, but since we do not know  $V$  this description is incomplete. Therefore we substitute the expression for  $\hat{u}$  in (1.28) into the PDE (1.27), giving us the PDE

$$\frac{\partial V}{\partial t}(t, x) + F^{\hat{\mathbf{u}}}(t, x) + \mathcal{A}^{\hat{\mathbf{u}}} V(t, x) = 0, \quad (1.29)$$

$$V(T, x) = \Phi(x). \quad (1.30)$$

5. Now we solve the PDE above! (See the remark below.) Then we put the solution  $V$  into expression (1.28). Using the verification theorem 1.3.2 we can now identify  $V$  as the optimal value function, and  $\hat{u}$  as the optimal control law.

**Remark 1.4.1** *The hard work of dynamic programming consists in solving the highly nonlinear PDE in step 5 above. There are of course no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed. In an actual case one usually tries to **guess** a solution, i.e. we typically make an **ansatz** for  $V$ , parameterized by a finite number of parameters, and then we use the PDE in order to identify the parameters. The making of an ansatz is often helped by the intuitive observation that if there is an analytical solution to the problem, then it seems likely that  $V$  inherits some structural properties from the boundary function  $\Phi$  as well as from the instantaneous utility function  $F$ .*

*For a general problem there is thus very little hope of obtaining an analytic solution, and it is worth pointing out that many of the known solved control problems have, to some extent, been “rigged” in order to be analytically solvable.*

## 1.5 Optimal Consumption and Investment

### 1.5.1 A Generalization

In many concrete applications, in particular in economics, it is natural to consider an optimal control problem, where the state variable is constrained to stay within a prespecified domain. As an example it may be reasonable to demand that the wealth of an investor is never allowed to become negative. We will now generalize our class of optimal control problems to allow for such considerations.

Let us therefore consider the following controlled SDE

$$dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t, \quad (1.31)$$

$$X_0 = x_0, \quad (1.32)$$

where as before we impose the control constraint  $u_t \in U$ . We also consider as given a fixed time interval  $[0, T]$ , and a fixed domain  $D \subseteq [0, T] \times R^n$ , and the basic idea is that when the state process hits the boundary  $\partial D$  of  $D$ , then the activity is at an end. It is thus natural to define the **stopping time**  $\tau$  by

$$\tau = \inf \{t \geq 0 \mid (t, X_t) \in \partial D\} \wedge T,$$

where  $x \wedge y = \min[x, y]$ . We consider as given an instantaneous utility function  $F(t, x, u)$  and a “bequest function”  $\Phi(t, x)$ , i.e. a mapping  $\Phi : \partial D \rightarrow R$ . The control problem to be considered is that of maximizing

$$E \left[ \int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi(\tau, X_\tau^{\mathbf{u}}) \right]. \quad (1.33)$$

In order for this problem to be interesting we have to demand that  $X_0 \in D$ , and the interpretation is that when we hit the boundary  $\partial D$ , the game is over and we obtain the bequest  $\Phi(\tau, X_\tau)$ . We see immediately that our earlier situation corresponds to the case when  $D = [0, T] \times R^n$  and when  $\Phi$  is constant in the  $t$ -variable.

In order to analyze our present problem we may proceed as in the previous sections, introducing the value function and the optimal value function exactly as before. The only new technical problem encountered is that of considering a stochastic integral with a stochastic limit of integration. Since this will take us outside the scope of the present text we will confine ourselves to giving the results. The proofs are (modulo the technicalities mentioned above) exactly as before.

**Theorem 1.5.1 (HJB equation)** *Assume that*

- *The optimal value function  $V$  is in  $C^{1,2}$ .*
- *An optimal law  $\hat{\mathbf{u}}$  exists.*

*Then the following hold.*



1.  $V$  satisfies the HJB equation

$$\begin{cases} \frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0, & \forall (t, x) \in D \\ V(t, x) = \Phi(t, x), & \forall (t, x) \in \partial D. \end{cases}$$

2. For each  $(t, x) \in D$  the supremum in the HJB equation above is attained by  $u = \hat{u}(t, x)$ .

**Theorem 1.5.2 (Verification theorem)** Suppose that we have two functions  $H(t, x)$  and  $g(t, x)$ , such that

- $H$  is sufficiently integrable, and solves the HJB equation

$$\begin{cases} \frac{\partial H}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\} = 0, & \forall (t, x) \in D \\ H(t, x) = \Phi(t, x), & \forall (t, x) \in \partial D. \end{cases}$$

- The function  $g$  is an admissible control law.
- For each fixed  $(t, x)$ , the supremum in the expression

$$\sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\}$$

is attained by the choice  $u = g(t, x)$ .

Then the following hold.

1. The optimal value function  $V$  to the control problem is given by

$$V(t, x) = H(t, x).$$

2. There exists an optimal control law  $\hat{u}$ , and in fact  $\hat{u}(t, x) = g(t, x)$ .

## 1.5.2 Optimal Consumption

In order to illustrate the technique we will now go back to the optimal consumption problem at the beginning of the chapter. We thus consider the problem of maximizing

$$E \left[ \int_0^T F(t, c_t) dt + \Phi(X_T) \right], \quad (1.34)$$

given the wealth dynamics

$$dX_t = X_t [u_t^0 r + u_t^1 \alpha] dt - c_t dt + u_t^1 \sigma X_t dW_t. \quad (1.35)$$

As usual we impose the control constraints

$$\begin{aligned} c_t &\geq 0, \quad \forall t \geq 0, \\ u_t^0 + u_t^1 &= 1, \quad \forall t \geq 0. \end{aligned}$$

In a control problem of this kind it is important to be aware of the fact that one may quite easily formulate a nonsensical problem. To take a simple example, suppose that we have  $\Phi = 0$ , and suppose that  $F$  is increasing and unbounded in the  $c$ -variable. Then the problem above degenerates completely. It does not possess an optimal solution at all, and the reason is of course that the consumer can increase his/her utility to any given level by simply consuming an arbitrarily large amount at every  $t$ . The consequence of this hedonistic behavior is of course the fact that the wealth process will, with very high probability, become negative, but this is neither prohibited by the control constraints, nor punished by any bequest function.

An elegant way out of this dilemma is to choose the domain  $D$  of the preceding section as  $D = [0, T] \times \{x \mid x > 0\}$ . With  $\tau$  defined as above this means, in concrete terms, that

$$\tau = \inf \{t > 0 \mid X_t = 0\} \wedge T.$$

A natural objective function in this case is thus given by

$$E \left[ \int_0^\tau F(t, c_t) dt \right], \quad (1.36)$$

which automatically ensures that when the consumer has no wealth, then all activity is terminated.

We will now analyze this problem in some detail. Firstly we notice that we can get rid of the constraint  $u_t^0 + u_t^1 = 1$  by defining a new control variable  $w$  as  $w = u^1$ , and then substituting  $1 - w$  for  $u^0$ . This gives us the state dynamics

$$dX_t = w_t [\alpha - r] X_t dt + (rX_t - c_t) dt + w_t \sigma X_t dW_t, \quad (1.37)$$

and the corresponding HJB equation is

$$\begin{cases} \frac{\partial V}{\partial t} + \sup_{c \geq 0, w \in R} \left\{ F(t, c) + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2} \right\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0. \end{cases}$$

We now specialize our example to the case when  $F$  is of the form

$$F(t, c) = e^{-\delta t} c^\gamma,$$

where  $0 < \gamma < 1$ . The economic reasoning behind this is that we now have an infinite marginal utility at  $c = 0$ . This will force the optimal consumption plan to be positive throughout the planning period, a fact which will facilitate

the analytical treatment of the problem. In terms of Remark 1.4.1 we are thus “rigging” the problem.

The static optimization problem to be solved w.r.t.  $c$  and  $w$  is thus that of maximizing

$$e^{-\delta t} c^\gamma + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

and, assuming an interior solution, the first order conditions are

$$\gamma c^{\gamma-1} = e^{\delta t} V_x, \quad (1.38)$$

$$w = \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, \quad (1.39)$$

where we have used subscripts to denote partial derivatives.

We again see that in order to implement the optimal consumption–investment plan (1.38)–(1.39) we need to know the optimal value function  $V$ . We therefore suggest a trial solution (see Remark 1.4.1), and in view of the shape of the instantaneous utility function it is natural to try a  $V$ -function of the form

$$V(t, x) = e^{-\delta t} h(t) x^\gamma, \quad (1.40)$$

where, because of the boundary conditions, we must demand that

$$h(T) = 0. \quad (1.41)$$

Given a  $V$  of this form we have (using  $\dot{\cdot}$  to denote the time derivative)

$$\frac{\partial V}{\partial t} = e^{-\delta t} \dot{h} x^\gamma - \delta e^{-\delta t} h x^\gamma, \quad (1.42)$$

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t} h x^{\gamma-1}, \quad (1.43)$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma(\gamma - 1) e^{-\delta t} h x^{\gamma-2}. \quad (1.44)$$

Inserting these expressions into (1.38)–(1.39) we get

$$\hat{\mathbf{w}}(t, x) = \frac{\alpha - r}{\sigma^2(1 - \gamma)}, \quad (1.45)$$

$$\hat{\mathbf{c}}(t, x) = x h(t)^{-1/(1-\gamma)}. \quad (1.46)$$

This looks very promising: we see that the candidate optimal portfolio is constant and that the candidate optimal consumption rule is linear in the wealth variable. In order to use the verification theorem we now want to show that a  $V$ -function of the form (1.40) actually solves the HJB equation. We therefore substitute the expressions (1.42)–(1.46) into the HJB equation. This gives us the equation

$$x^\gamma \left\{ \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} \right\} = 0,$$

where the constants  $A$  and  $B$  are given by

$$\begin{aligned} A &= \frac{\gamma(\alpha - r)^2}{\sigma^2(1 - \gamma)} + r\gamma - \frac{1}{2} \frac{\gamma(\alpha - r)^2}{\sigma^2(1 - \gamma)} - \delta \\ B &= 1 - \gamma. \end{aligned}$$

If this equation is to hold for all  $x$  and all  $t$ , then we see that  $h$  must solve the ODE

$$\dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} = 0, \quad (1.47)$$

$$h(T) = 0. \quad (1.48)$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly (see the exercises).

Summing up, we have shown that if we define  $V$  as in (1.40) with  $h$  defined as the solution to (1.47)–(1.48), and if we define  $\hat{w}$  and  $\hat{c}$  by (1.45)–(1.46), then  $V$  satisfies the HJB equation, and  $\hat{w}$ ,  $\hat{c}$  attain the supremum in the equation. The verification theorem then tells us that we have indeed found the optimal solution.

## 1.6 The Mutual Fund Theorems

In this section we will briefly go through the “Merton mutual fund theorems”, originally presented in Merton (1971).

### 1.6.1 The Case with No Risk Free Asset

We consider a financial market with  $n$  asset prices  $S_1, \dots, S_n$ . To start with we do not assume the existence of a risk free asset, and we assume that the price vector process  $S(t)$  has the following dynamics under the objective measure  $P$ .

$$dS = D(S)\alpha dt + D(S)\sigma dW. \quad (1.49)$$

Here  $W$  is a  $k$ -dimensional standard Wiener process,  $\alpha$  is an  $n$ -vector,  $\sigma$  is an  $n \times k$  matrix, and  $D(S)$  is the diagonal matrix

$$D(S) = \text{diag}[S_1, \dots, S_n].$$

In more pedestrian terms this means that

$$dS_i = S_i\alpha_i dt + S_i\sigma_i dW,$$

where  $\sigma_i$  is the  $i$ th row of the matrix  $\sigma$ .

We denote the investment strategy (relative portfolio) by  $w$ , and the consumption plan by  $c$ . If the pair  $(w, c)$  is self-financing, then it follows from the  $S$ -dynamics above, and from Lemma ??, that the dynamics of the wealth process  $X$  are given by

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW. \quad (1.50)$$

We also take as given an instantaneous utility function  $F(t, c)$ , and we basically want to maximize

$$E \left[ \int_0^T F(t, c_t) dt \right]$$

where  $T$  is some given time horizon. In order not to formulate a degenerate problem we also impose the condition that wealth is not allowed to become negative, and as before this is dealt with by introducing the stopping time

$$\tau = \inf \{t > 0 \mid X_t = 0\} \wedge T.$$

Our formal problem is then that of maximizing

$$E \left[ \int_0^\tau F(t, c_t) dt \right]$$

given the dynamics (1.49)–(1.50), and subject to the control constraints

$$\sum_1^n w_i = 1, \tag{1.51}$$

$$c \geq 0. \tag{1.52}$$

Instead of (1.51) it is convenient to write

$$e'w = 1,$$

where  $e$  is the vector in  $R^n$  which has the number 1 in all components, i.e.  $e' = (1, \dots, 1)$ .

The HJB equation for this problem now becomes

$$\begin{cases} \frac{\partial V}{\partial t}(t, x, s) + \sup_{e'w=1, c \geq 0} \{F(t, c) + \mathcal{A}^{c,w}V(t, x, s)\} = 0, \\ V(T, x, s) = 0, \\ V(t, 0, s) = 0. \end{cases}$$

In the general case, when the parameters  $\alpha$  and  $\sigma$  are allowed to be functions of the price vector process  $S$ , the term  $\mathcal{A}^{c,w}V(t, x, s)$  turns out to be rather forbidding (see Merton's original paper). It will in fact involve partial derivatives to the second order with respect to all the variables  $x, s_1, \dots, s_n$ .

If, however, we assume that  $\alpha$  and  $\sigma$  are deterministic and constant over time, then we see by inspection that the wealth process  $X$  is a Markov process, and since the price processes do not appear, neither in the objective function nor in the definition of the stopping time, we draw the conclusion that in this case  $X$  itself will act as the state process, and we may forget about the underlying  $S$ -process completely.

Under these assumptions we may thus write the optimal value function as  $V(t, x)$ , with no  $s$ -dependence, and after some easy calculations the term  $\mathcal{A}^{c,w}V$  turns out to be

$$\mathcal{A}^{c,w}V = xw'\alpha \frac{\partial V}{\partial x} - c \frac{\partial V}{\partial x} + \frac{1}{2}x^2w'\Sigma w \frac{\partial^2 V}{\partial x^2},$$

where the matrix  $\Sigma$  is given by

$$\Sigma = \sigma\sigma'.$$

We now summarize our assumptions.

**Assumption 1.6.1** *We assume that*

- *The vector  $\alpha$  is constant and deterministic.*
- *The matrix  $\sigma$  is constant and deterministic.*
- *The matrix  $\sigma$  has rank  $n$ , and in particular the matrix  $\Sigma = \sigma\sigma'$  is positive definite and invertible.*

We note that, in terms of contingent claims analysis, the last assumption means that the market is complete. Denoting partial derivatives by subscripts we now have the following HJB equation

$$\begin{cases} V_t(t, x) + \sup_{w'e=1, c \geq 0} \left\{ F(t, c) + (xw'\alpha - c)V_x(t, x) + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) \right\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0. \end{cases}$$

If we relax the constraint  $w'e = 1$ , the Lagrange function for the static optimization problem is given by

$$L = F(t, c) + (xw'\alpha - c)V_x(t, x) + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) + \lambda(1 - w'e).$$

Assuming the problem to be regular enough for an interior solution we see that the first order condition for  $c$  is

$$\frac{\partial F}{\partial c}(t, c) = V_x(t, x).$$

The first order condition for  $w$  is

$$x\alpha'V_x + x^2V_{xx}w'\Sigma = \lambda e',$$

so we can solve for  $w$  in order to obtain

$$\hat{w} = \Sigma^{-1} \left[ \frac{\lambda}{x^2V_{xx}}e - \frac{xV_x}{x^2V_{xx}}\alpha \right]. \quad (1.53)$$

Using the relation  $e'w = 1$  this gives  $\lambda$  as

$$\lambda = \frac{x^2V_{xx} + xV_xe'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

and inserting this into (1.53) gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \frac{V_x}{xV_{xx}}\Sigma^{-1} \left[ \frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha \right]. \quad (1.54)$$

To see more clearly what is going on we can write this expression as

$$\hat{w}(t) = g + Y(t)h, \quad (1.55)$$

where the fixed vectors  $g$  and  $h$  are given by

$$g = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \quad (1.56)$$

$$h = \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \quad (1.57)$$

whereas  $Y$  is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}. \quad (1.58)$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional “optimal portfolio line”

$$g + sh,$$

in the  $(n - 1)$ -dimensional “portfolio hyperplane”  $\Delta$ , where

$$\Delta = \{w \in R^n \mid e'w = 1\}.$$

We now make the obvious geometric observation that if we fix two points on the optimal portfolio line, say the points  $w^a = g + ah$  and  $w^b = g + bh$ , then any point  $w$  on the line can be written as an affine combination of the basis points  $w^a$  and  $w^b$ . An easy calculation shows that if  $w^s = g + sh$  then we can write

$$w^s = \mu w^a + (1 - \mu)w^b,$$

where

$$\mu = \frac{s - b}{a - b}.$$

The point of all this is that we now have an interesting economic interpretation of the optimality results above. Let us thus fix  $w^a$  and  $w^b$  as above on the optimal portfolio line. Since these points are in the portfolio plane  $\Delta$  we can interpret them as the relative portfolios of two fixed mutual funds. We may then write (1.55) as

$$\hat{w}(t) = \mu(t)w^a + (1 - \mu(t))w^b, \quad (1.59)$$

with

$$\mu(t) = \frac{Y(t) - b}{a - b}.$$

Thus we see that the optimal portfolio  $\hat{w}$  can be obtained as a “super portfolio” where we allocate resources between two fixed mutual funds.

**Theorem 1.6.1 (Mutual fund theorem)** *Assume that the problem is regular enough to allow for an interior solution. Then there exists a one-dimensional parameterized family of mutual funds, given by  $w^s = g + sh$ , where  $g$  and  $h$  are defined by (1.56)–(1.57), such that the following hold.*

1. For each fixed  $s$  the relative portfolio  $w^s$  stays fixed over time.
2. For any fixed choice of  $a \neq b$  the optimal portfolio  $\hat{w}(t)$  is, for all values of  $t$ , obtained by allocating all resources between the fixed funds  $w^a$  and  $w^b$ , i.e.

$$\begin{aligned}\hat{w}(t) &= \mu^a(t)w^a + \mu^b(t)w^b, \\ \mu^a(t) + \mu^b(t) &= 1.\end{aligned}$$

3. The relative proportions  $(\mu^a, \mu^b)$  of the portfolio wealth allocated to  $w^a$  and  $w^b$  respectively are given by

$$\begin{aligned}\mu^a(t) &= \frac{Y(t) - b}{a - b}, \\ \mu^b(t) &= \frac{a - Y(t)}{a - b},\end{aligned}$$

where  $Y$  is given by (1.58).

### 1.6.2 The Case with a Risk Free Asset

Again we consider the model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t), \quad (1.60)$$

with the same assumptions as in the preceding section. We now also take as given the standard risk free asset  $B$  with dynamics

$$dB = rBdt.$$

Formally we can denote this as a new asset by subscript zero, i.e.  $B = S_0$ , and then we can consider relative portfolios of the form  $w = (w_0, w_1, \dots, w_n)'$  where of course  $\sum_0^n w_i = 1$ . Since  $B$  will play such a special role it will, however, be convenient to eliminate  $w_0$  by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

and then use the letter  $w$  to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \dots, w_n)',$$

and we note that this truncated portfolio vector is allowed to take any value in  $R^n$ .

Given this notation it is easily seen that the dynamics of a self-financing portfolio are given by

$$dX = X \cdot \left\{ \sum_1^n w_i \alpha_i + \left( 1 - \sum_1^n w_i \right) r \right\} dt - cdt + X \cdot w' \sigma dW.$$



That is,

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW, \quad (1.61)$$

where as before  $e \in R^n$  denotes the vector  $(1, 1, \dots, 1)'$ .

The HJB equation now becomes

$$\begin{cases} V_t(t, x) + \sup_{c \geq 0, w \in R^n} \{F(t, c) + \mathcal{A}^{c, w}V(t, x)\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0, \end{cases}$$

where

$$\mathcal{A}^c V = xw'(\alpha - re)V_x(t, x) + (rx - c)V_x(t, x) + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x).$$

The first order conditions for the static optimization problem are

$$\begin{aligned} \frac{\partial F}{\partial c}(t, c) &= V_x(t, x), \\ \hat{w} &= -\frac{V_x}{xV_{xx}}\Sigma^{-1}(\alpha - re), \end{aligned}$$

and again we have a geometrically obvious economic interpretation.

**Theorem 1.6.2 (Mutual fund theorem)** *Given assumptions as above, the following hold.*

1. *The optimal portfolio consists of an allocation between two fixed mutual funds  $w^0$  and  $w^f$ .*
2. *The fund  $w^0$  consists only of the risk free asset.*
3. *The fund  $w^f$  consists only of the risky assets, and is given by*

$$w^f = \Sigma^{-1}(\alpha - re).$$

4. *At each  $t$  the optimal relative allocation of wealth between the funds is given by*

$$\begin{aligned} \mu^f(t) &= -\frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}, \\ \mu^0(t) &= 1 - \mu^f(t). \end{aligned}$$

Note that this result is not a corollary of the corresponding result from the previous section. Firstly it was an essential ingredient in the previous results that the volatility matrix of the price vector was invertible. In the case with a riskless asset the volatility matrix for the entire price vector  $(B, S_1, \dots, S_n)$  is of course degenerate, since its first row (having subscript zero) is identically equal to zero. Secondly, even if one assumes the results from the previous section, i.e. that the optimal portfolio is built up from two fixed portfolios, it is not at all obvious that one of these basis portfolios can be chosen so as to consist of the risk free asset alone.

## 1.7 Exercises

**Exercise 1.1** Solve the problem of maximizing logarithmic utility

$$E \left[ \int_0^T e^{-\delta t} \ln(c_t) dt + K \cdot \ln(X_T) \right],$$

given the usual wealth dynamics

$$dX_t = X_t [u_t^0 r + u_t^1 \alpha] dt - c_t dt + u^1 \sigma X_t dW_t,$$

and the usual control constraints

$$\begin{aligned} c_t &\geq 0, \quad \forall t \geq 0, \\ u_t^0 + u_t^1 &= 1, \quad \forall t \geq 0. \end{aligned}$$

**Exercise 1.2** A Bernoulli equation is an ODE of the form

$$\dot{x}_t + A_t x_t + B_t x_t^\alpha = 0,$$

where  $A$  and  $B$  are deterministic functions of time and  $\alpha$  is a constant.

If  $\alpha = 1$  this is a linear equation, and can thus easily be solved. Now consider the case  $\alpha \neq 1$  and introduce the new variable  $y$  by

$$y_t = x_t^{1-\alpha}.$$

Show that  $y$  satisfies the **linear** equation

$$\dot{y}_t + (1 - \alpha)A_t y_t + (1 - \alpha)B_t = 0.$$

**Exercise 1.3** Use the previous exercise in order to solve (1.47)–(1.48) explicitly.

**Exercise 1.4** Consider as before state process dynamics

$$dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t$$

and the usual restrictions for  $u$ . Our entire derivation of the HJB equation has so far been based on the fact that the objective function is of the form

$$\int_0^T F(t, X_t, u_t) dt + \Phi(X_T).$$

Sometimes it is natural to consider other criteria, like the expected **exponential utility** criterion

$$E \left[ \exp \left\{ \int_0^T F(t, X_t, u_t) dt + \Phi(X_T) \right\} \right].$$

For this case we define the optimal value function as the supremum of

$$E_{t,x} \left[ \exp \left\{ \int_t^T F(s, X_s, u_s) dt + \Phi(X_T) \right\} \right].$$

Follow the reasoning in Section 1.3 in order to show that the HJB equation for the expected exponential utility criterion is given by

$$\begin{cases} \frac{\partial V}{\partial t}(t, x) + \sup_u \{V(t, x)F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0, \\ V(T, x) = e^{\Phi(x)}. \end{cases}$$

**Exercise 1.5** Solve the problem to minimize

$$E \left[ \exp \left\{ \int_0^T u_t^2 dt + X_T^2 \right\} \right]$$

given the scalar dynamics

$$dX = (ax + u)dt + \sigma dW$$

where the control  $u$  is scalar and there are no control constraints.

**Hint:** Make the ansatz

$$V(t, x) = e^{A(t)x^2 + B(t)}.$$

**Exercise 1.6** Study the general linear-exponential-quadratic control problem of minimizing

$$E \left[ \exp \left\{ \int_0^T \{X_t' Q X_t + u_t' R u_t\} dt + X_T' H X_T \right\} \right]$$

given the dynamics

$$dX_t = \{A X_t + B u_t\} dt + C dW_t.$$

**Exercise 1.7** The object of this exercise is to connect optimal control to martingale theory. Consider therefore a general control problem of minimizing

$$E \left[ \int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dt + \Phi(X_T^{\mathbf{u}}) \right]$$

given the dynamics

$$dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t,$$

and the constraints

$$\mathbf{u}(t, x) \in U.$$

Now, for any control law  $\mathbf{u}$ , define the **total cost process**  $C(t; \mathbf{u})$  by

$$C(t; \mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + E_{t, X_t^{\mathbf{u}}} \left[ \int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) dt + \Phi(X_T^{\mathbf{u}}) \right],$$

*i.e.*

$$C(t; \mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \mathcal{J}(t, X_t^{\mathbf{u}}, \mathbf{u}).$$

Use the HJB equation in order to prove the following claims.

- (a) If  $\mathbf{u}$  is an arbitrary control law, then  $C$  is a submartingale.
- (b) If  $\mathbf{u}$  is optimal, then  $C$  is a martingale.

## 1.8 Notes

Standard references on optimal control are [14] and [27]. A very clear exposition can be found in [31]. For more recent work, using viscosity solutions, see [15]. The classical papers on optimal consumption are [29] and [30]. For optimal trading under constraints, and its relation to derivative pricing see [7] and references therein.

## Chapter 2

# The Martingale Approach to Optimal Investment

In Chapter 1 we studied optimal investment and consumption problems, using dynamic programming. This approach transforms the original stochastic optimal control problem into the problem of solving a non linear deterministic PDE, namely the Hamilton-Jacobi-Bellman equation, so the probabilistic nature of the problem disappears as soon as we have formulated the HJB equation.

In this chapter we will present an alternative method of solving optimal investment problems. This method is commonly referred to as "the martingale approach" and it has the advantage that it is in some sense more direct and more probabilistic than dynamic programming, and we do not need to assume a Markovian structure. It should be noted however, that while dynamic programming can be applied to **any** Markovian stochastic optimal control problem, the martingale approach is only applicable to financial portfolio problems, and in order to get explicit results we also typically need to assume market completeness.

### 2.1 Generalities

We consider a financial market living on a stochastic basis  $(\Omega, \mathcal{F}, \mathbf{F}, P)$ , where  $P$  is the objective probability measure. The basis carries an  $n$ -dimensional  $P$ -Wiener process  $W$ , and the filtration  $\mathbf{F}$  is the one generated by the  $W$  process so  $\mathbf{F} = \mathbf{F}^W$ .

The financial market under consideration consists of  $n$  non-dividend paying risky assets ("stocks") with price processes  $S^1, \dots, S^n$ , and a bank account with price process  $B$ . The formal assumptions concerning the price dynamics are as follows.

**Assumption 2.1.1** *We assume the following.*

- The risky asset prices have  $P$ -dynamics given by

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sigma_t^i dW_t, \quad i = 1, \dots, n. \quad (2.1)$$

Here  $\alpha^1, \dots, \alpha^n$  are assumed to be  $\mathbf{F}$ -adapted scalar processes, and  $\sigma^1, \dots, \sigma^n$  are  $\mathbf{F}^W$ -adapted  $d$ -dimensional row vector processes.

- The short rate  $r$  is assumed to be constant, i.e. the bank account has dynamics given by

$$dB_t = rB_t dt.$$

**Remark 2.1.1** Note that we do not make any Markovian assumptions, so in particular the process  $\alpha$  and  $\sigma$  are allowed to be arbitrary adapted path dependent processes. Of particular interest is of course the Markovian case i.e. when  $\alpha$  and  $\sigma$  are deterministic functions of  $t$  and  $S_t$  so  $\alpha_t = \alpha_t(S_t)$  and  $\sigma_t = \sigma_t(S_t)$ .

Defining the stock vector process  $S$  by

$$S = \begin{pmatrix} S^1 \\ \vdots \\ S^n \end{pmatrix},$$

the rate of return vector process  $\alpha$  by

$$\alpha = \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix},$$

and the volatility matrix  $\sigma$  by

$$\sigma = \begin{pmatrix} -\sigma^1 - \\ \vdots \\ -\sigma^n - \end{pmatrix},$$

we can write the stock price dynamics as

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

where  $D(S)$  denotes the diagonal matrix with  $S^1, \dots, S^n$  on the diagonal.

We will need an important assumption concerning the volatility matrix.

**Assumption 2.1.2** We assume that with probability one the volatility matrix  $\sigma(t)$  is non-singular for all  $t$ .

The point of this assumption is the following result, the proof of which is obvious.

**Proposition 2.1.1** Under the assumptions above, the market model is complete.

## 2.2 The Basic Idea

Let us consider an investor with initial capital  $x$  and a utility function  $U$  for terminal wealth. For any self financing portfolio, we denote the corresponding portfolio value process by  $X$  and our problem is to maximize expected utility

$$E^P [U(X_T)],$$

over the class of self financing adapted portfolios with the initial condition  $X(0) = x$ .

In Chapter 1 we viewed this as a **dynamic** optimization problem and attacked it by using dynamic programming. A different way of formulating the problem is however as follows. Define  $\mathcal{K}_T$  as the set of contingent  $T$ -claims which can be replicated by a self financing portfolio with initial capital  $x$ . Then our basic problem can be formulated as the **static** problem

$$\max_{X_T} E^P [U(X_T)]$$

subject to the **static** constraint

$$X_T \in \mathcal{K}_T.$$

In this formulation, the focus is not on the optimal portfolio strategy but instead on the terminal wealth  $X_T$ . We now have the following important observation, which follows immediately from the market completeness.

**Proposition 2.2.1** *With assumptions as above, the following conditions are equivalent for any random variable  $X_T \in \mathcal{F}_T$ .*

$$X_T \in \mathcal{K}_T. \tag{2.2}$$

$$e^{-rT} E^Q [X_T] = x. \tag{2.3}$$

The implication of this simple observation is that we can now **decouple** the problem of determining the optimal terminal wealth profile from the problem of determining the optimal portfolio. Schematically we proceed as follows.

- Instead of solving the dynamic control problem, we solve the **static** problem

$$\max_{X_T \in \mathcal{F}_T} E^P [U(X_T)] \tag{2.4}$$

subject to the budget constraint

$$e^{-rT} E^Q [X_T] = x, \tag{2.5}$$

where  $x$  is the initial wealth, and  $Q$  is the unique martingale measure.

- Given the optimal wealth profile  $\hat{X}_T$ , we can (in principle) compute the corresponding generating portfolio using martingale representation results.

### 2.3 The Optimal Terminal Wealth

The static problem (2.4) with the constraint (2.5) can easily be solved using Lagrange relaxation. We start by rewriting the budget constraint (2.5) as

$$e^{-rT} E^P [L_T X] = x,$$

where  $L$  is the likelihood process between  $P$  and  $Q$ , i.e.,

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t.$$

We now relax the budget constraint to obtain the Lagrangian

$$\mathcal{L} = E^P [U(X)] - \lambda (e^{-rT} E^P [L_T X] - x),$$

so

$$\mathcal{L} = \int_{\Omega} \{U(X(\omega)) - \lambda [e^{-rT} L_T(\omega) X(\omega) - x]\} dP(\omega).$$

It now remains to maximize the unconstrained Lagrangian over  $X_T$ , but this is trivial: Since we have no constraints we can maximize  $\mathcal{L}$  for each  $\omega$ . The optimality condition is

$$U'(X_T) = \lambda e^{-rT} L_T$$

so the optimal wealth profile is given by

$$\hat{X}_T = F(\lambda M_T), \tag{2.6}$$

where  $M$  is the stochastic discount factor defined as usual by

$$M_t = B_t^{-1} L_t,$$

and  $F$  is the functional inverse of the utility function  $U$ , so  $F = (U')^{-1}$ .

We do in fact have an explicit expression for the Radon-Nikodym derivative  $L_T$  above. From the price dynamics (2.1) and the Girsanov Theorem it is easily seen that the  $L$  dynamics are given by

$$\begin{aligned} dL_t &= \{\sigma_t^{-1} (\mathbf{r} - \alpha_t)\}' dW_t, \\ L_0 &= 1, \end{aligned}$$

where  $'$  denotes transpose, and  $\mathbf{r}$  denotes the  $n$ -column vector with  $r$  in each component. We thus have the explicit formula

$$L_t = \exp \left\{ \int_0^t \{\sigma_s^{-1} (\mathbf{r} - \alpha_s)\}' dW_s - \frac{1}{2} \int_0^t \|\sigma_s^{-1} (\mathbf{r} - \alpha_s)\|^2 ds \right\}, \tag{2.7}$$

where  $'$  denotes transpose. We collect our results in a proposition.

**Proposition 2.3.1** *Under the assumptions above, the optimal terminal wealth profile  $\hat{X}_T$  is given by*

$$\hat{X}_T = F(\lambda M_T). \tag{2.8}$$

*The stochastic discount factor is defined by  $M_t = B_t^{-1} L_t$ , the likelihood process  $L$  is given by (2.7), and the Lagrange multiplier  $\lambda$  is determined by the budget constraint (2.5). The function  $F$  is the inverse of the marginal utility function  $U'$ .*



## 2.4 The Optimal Portfolio

In the previous section we saw that we could, in principle quite easily, derive a closed form expression for the optimal terminal wealth  $\hat{X}_T$ . The next step is to determine the optimal portfolio strategy, i.e. the portfolio which generates  $\hat{X}_T$ . The general idea for how to do this is in fact quite simple, although it may be difficult to carry it out in a concrete case. It relies on using the martingale representation Theorem and works as follows.

If we denote the vector of relative portfolio weights on the  $n$  risky assets by  $u_t = (u_t^1, \dots, u_t^n)$ , then it is easy to see that the dynamics of the induced wealth process  $X$  are given by

$$dX_t = X_t u_t \alpha_t dt + X_t (1 - u_t \mathbf{1}) r dt + X_t u_t \sigma_t dW_t,$$

where  $\mathbf{1}$  denotes the column vector in  $R^n$  with 1 in every position. If we now define the normalized process  $Z$  as

$$Z_t = \frac{X_t}{B_t} = e^{-rt} X_t,$$

then we know from general theory that  $Z$  is a  $Q$  martingale. From the Itô formula we have

$$dZ_t = Z_t u_t \{\alpha_t - r \mathbf{1}\} dt + Z_t u_t \sigma_t dW_t,$$

and, since we know that the diffusion part is unchanged under a Girsanov transformation, the  $Q$  dynamics of  $Z$  are

$$dZ_t = Z_t u_t \sigma_t dW_t^Q, \tag{2.9}$$

where  $W^Q$  is a  $Q$ -Wiener process. We can now proceed along the following lines, where  $\hat{X}_T$  is given by Proposition 2.3.1.

1. Define the process  $Z$  by

$$Z_t = E^Q \left[ e^{-rT} \hat{X}_T \mid \mathcal{F}_t \right].$$

2. Since  $Z$  is a  $Q$  martingale it follows from the Martingale Representation Theorem that  $Z$  has dynamics of the form

$$dZ_t = h_t dW_t^Q, \tag{2.10}$$

for some adapted process  $h$ .

3. Comparing (2.10) with (2.9) we can determine the portfolio strategy  $u$ , which generates  $\hat{X}_T$  by solving the equation

$$Z_t u_t \sigma_t = h_t,$$

for every  $t$ . Since  $\sigma_t$  was assumed to be invertible for every  $t$ , this is easily done and we can now collect our results.

**Proposition 2.4.1** *The vector process  $\hat{u}$  of optimal portfolio weights on the risky assets is given by*

$$\hat{u}_t = \frac{1}{Z_t} h_t \sigma_t^{-1}, \quad (2.11)$$

where  $h$  is given, through the Martingale Representation Theorem, by (2.10).

We see that we can "in principle" determine the optimal portfolio strategy  $\hat{u}$ . For a concrete model, the result of Proposition 2.4.1 does, however, not lead directly to a closed form expression for  $\hat{u}$ . The reason is that the formula (2.11) involves the process  $h$  which is provided by the martingale representation theorem. That theorem, however, is an **existence** theorem, so we know that  $h$  exists, but we typically do not know what it looks like. To obtain closed form expressions, we therefore have to make some further model assumptions, and typically we will have to assume a Markovian structure. In the next chapter we will study the problem of determining the optimal portfolio in some detail, but for the moment we will be content to exemplify by studying the simple case of log utility.

## 2.5 Log utility

The simplest of all utility functions is log utility where the utility function is given by

$$U(x) = \ln(x).$$

This implies that

$$F(y) = \frac{1}{y}.$$

From (2.8) we thus see that the optimal terminal wealth is given by the expression

$$\hat{X}_T = \frac{1}{\lambda} M_T^{-1}.$$

The Lagrange multiplier is easily determined from the budget constraint

$$E^P \left[ M_T \cdot \hat{X}_T \right] = x,$$

so we obtain

$$\hat{X}_T = x M_T^{-1}.$$

We can in fact compute, not only the terminal optimal wealth, but also the entire optimal portfolio process. From risk neutral valuation we have

$$\hat{X}_t = \frac{1}{M_t} E^P \left[ M_T \cdot \hat{X}_T \mid \mathcal{F}_t \right],$$

which immediately gives us the optimal portfolio process as

$$\hat{X}_t = x M_t^{-1}.$$

We have thus proved the following result.

**Proposition 2.5.1** *For log utility, and without any assumptions on the asset price dynamics, the optimal terminal wealth is given by*

$$\hat{X}_T = xM_T^{-1}, \quad (2.12)$$

and the optimal portfolio process is given by

$$\hat{X}_t = xM_t^{-1}. \quad (2.13)$$

This implies that optimal portfolio for log utility is **myopic** in the sense that the value process  $\hat{X}_t = xM_t^{-1}$  which is optimal for maximizing log utility with time horizon  $T$ , is in fact also optimal for maximizing log utility for **every** time horizon.

We now go on to determine the optimal portfolio, and for simplicity we restrict ourselves to the scalar case.

**Assumption 2.5.1** *We assume that, under  $P$ , the stock price dynamics are given by*

$$dS_t = S_t\alpha_t dt + S_t\sigma_t dW_t, \quad (2.14)$$

where mean rate of return  $\alpha$  and the volatility  $\sigma$  are adapted processes. We denote the portfolio weight on  $S$  by  $u$ .

In order to determine the optimal portfolio weight  $\hat{u}$  we do as follows. From general theory we know that the  $\hat{X}$ -dynamics are of the form

$$\hat{X}_t = (\dots)dt + u_t\hat{X}_t\sigma_t dW_t$$

where we do not care about the  $dt$ -term.

We then recall that the dynamics of  $M$  are of the form

$$dM_t = -r_t M_t dt + \varphi_t M_t dW_t,$$

where the Girsanov kernel  $\varphi$  is given by

$$\varphi_t = \frac{r_t - \alpha_t}{\sigma_t}.$$

Using the  $M$  dynamics, and applying the Ito formula to (2.13), we obtain

$$d\hat{X}_t = (\dots)dt - \hat{X}_t\varphi_t dW_t.$$

Comparing the two expressions for  $d\hat{X}_t$  allows us to identify  $\hat{u}$ , and we have proved the following result.

**Proposition 2.5.2** *For log utility, the optimal portfolio weight on the risky asset is given by*

$$\hat{u}_t = \frac{\alpha_t - r_t}{\sigma_t^2}.$$

## 2.6 Exercises

**Exercise 2.1** *In this exercise we will see how intermediate consumption can be handled by the martingale approach. We make the assumptions of Section 2.1 and the problem is to maximize*

$$E^P \left[ \int_0^T g(s, c_s) ds + U(X_T) \right]$$

over the class of self financing portfolios with initial wealth  $x$ . Here  $c$  is the consumption rate for a consumption good with unit price, so  $c$  denotes consumption rate in terms of dollars per unit time. The function  $g$  is the local utility of consumption,  $U$  is utility of terminal wealth, and  $X$  is portfolio wealth.

(a) Convince yourself that the appropriate budget constraint is given by

$$E^Q \left[ \int_0^T e^{-rs} c_s ds + e^{-rT} X_T \right] = x.$$

(b) Show that the first order condition for the optimal terminal wealth and the optimal consumption plan are given by

$$\begin{aligned} \hat{c}_t &= G(\lambda e^{-rt} L_t), \\ \hat{X}_X &= F(\lambda e^{-rT} L_T), \end{aligned}$$

where  $G = (g')^{-1}$ ,  $F = (U')^{-1}$ ,  $L$  is the usual likelihood process, and  $\lambda$  is a Lagrange multiplier.

**Exercise 2.2** *Consider the setup in the previous exercise and assume that  $g(c) = \ln(c)$  and  $U(x) = a \ln(x)$ , where  $a$  is a positive constant. Compute the optimal consumption plan, and the optimal terminal wealth profile.*

**Exercise 2.3** *Consider the log-optimal portfolio given by Proposition 2.5.1 as*

$$X_t = e^{rt} x L_t^{-1}.$$

Show that this portfolio is the “ $P$  numeraire portfolio” in the sense that if  $\Pi$  is the arbitrage free price process for any asset in the economy, underlying or derivative, then the normalized asset price

$$\frac{\Pi_t}{X_t}$$

is a martingale under the objective probability measure  $P$ .

## 2.7 Notes

The basic papers for the martingale approach to optimal investment problems see [21] and [4] for the complete market case. The theory for the (much harder) incomplete market case were developed in [23], and [25]. A very readable overview of convex duality theory for the incomplete market case, containing an extensive bibliography, is given in [33].

## Chapter 3

# Connections between DynP and MG

In this chapter we study a rather simple consumption/investment model within a Black-Scholes framework. We then attack this problem using dynamic programming (henceforth DynP) as well as by the martingale approach (henceforth MG) and we investigate how these approaches are related. The results can be generalized to much more complicated settings and we will use them repeatedly in our study of equilibrium models.

### 3.1 The model

We consider a standard Black-Scholes model of the form

$$\begin{aligned}dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\dB_t &= rB_t dt\end{aligned}$$

and the problem is that of maximizing expected utility of the form

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right]$$

with the usual portfolio dynamics

$$dX_t = X_t u_t (\alpha - r) dt + (rX_t - c_t) dt + X_t u_t \sigma dW_t$$

where we have used the notation

$$\begin{aligned}X_t &= \text{portfolio value,} \\c_t &= \text{consumption rate,} \\u_t &= \text{weight on the risky asset.}\end{aligned}$$

We impose no restrictions on  $u$ , we require that  $c$  is nonnegative, and we handle non negativity of  $X$  in the usual way by introducing the stopping time

$$\tau = \inf \{t \geq 0 : X_t = 0\},$$

and replacing  $T$  by  $\tau$  in the integral above. We make the usual convexity assumptions about  $\Phi$  and  $U$ , and we also assume that the problem is “nice” in the sense that there exists an optimal solution satisfying the HJB equation, that the Verification Theorem is in force, and that the optimal consumption is interior.

### 3.2 Dynamic programming

The HJB equation for the optimal value function  $V(t, x)$  is given by

$$\begin{aligned} V_t + \sup_{(c,u)} \left\{ U(t, c) + xu(\alpha - r)V_x + (rx - c)V_x + \frac{1}{2}x^2u^2\sigma^2V_{xx} \right\} &= 0, \quad (3.1) \\ V(T, x) &= \Phi(x) \\ V(t, 0) &= 0. \end{aligned}$$

From the first order condition we obtain

$$U_c(t, c) = V_x(t, x), \quad (3.2)$$

$$\hat{u}(t, x) = -\frac{(\alpha - r)}{\sigma^2} \cdot \frac{V_x(t, x)}{xV_{xx}(t, x)}. \quad (3.3)$$

where  $c$  in the first equation really is  $\hat{c}(t, x)$ .

Plugging the expression for  $\hat{u}$  into the HJB equation gives us the PDE

$$V_t + U(t, \hat{c}) + (rx - \hat{c})V_x - \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0, \quad (3.4)$$

with the same boundary conditions as above.

Note that equation (3.4) is a **non-linear** PDE. The term  $V_x$  enters in a non-linear way in the expression for  $\hat{c}$  and, even disregarding that nonlinearity, we have the highly nonlinear term  $\frac{V_x^2}{V_{xx}}$ . Solving a linear PDE is hard enough and solving a non-linear PDE is of course even harder, so this is a matter of some concern. It is thus natural to ask whether it is possible to remove at least the second nonlinearity by a clever and/or natural change of variables.

### 3.3 The martingale approach

By applying the usual arguments we see that the original problem is equivalent to the problem of maximizing the expected utility

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right]$$

over all consumption processes  $c$  and terminal wealth profiles  $X_T$ , under the budget constraint

$$E^P \left[ \int_0^T e^{-rt} L_t c_t dt + e^{-rT} L_T X_T \right], \quad (3.5)$$

where  $L$  is the likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t,$$

with dynamics

$$\begin{cases} dL_t &= L_t \varphi_t dW_t, \\ L_0 &= 1 \end{cases}$$

and where the Girsanov kernel  $\varphi$  is given by

$$\varphi_t = \frac{r - \alpha}{\sigma}.$$

The Lagrangian for this problem is

$$E^P \left[ \int_0^T \{U(t, c_t) - \lambda e^{-rt} L_t c_t\} dt + \Phi(X_T) - e^{-rT} \lambda L_T X_T \right] + \lambda x_0$$

where  $\lambda$  is the Lagrange multiplier and  $x_0$  the initial wealth. The first order conditions are

$$U_c(t, c) = \lambda M_t, \quad (3.6)$$

$$\Phi'(X_T) = \lambda M_T. \quad (3.7)$$

where  $M$  denotes the stochastic discount factor (SDF), defined by

$$M_t = B_t^{-1} L_t.$$

Denoting the the inverse of  $\Phi'$  by  $F$  and the inverse (in the  $c$ -variable) of  $U_c$  by  $G$  we can write the optimality conditions on the form

$$\hat{c}_t = G(t, \lambda M_t), \quad (3.8)$$

$$\hat{X}_T = F(\lambda M_T). \quad (3.9)$$

We now have our first connection between the DynP and the MG approach.

**Proposition 3.3.1** *With notation as above we have*

$$V_x(t, \hat{X}_t) = \lambda M_t, \quad (3.10)$$

*In other words: Along the optimal trajectory, the indirect marginal utility is (up to a scaling factor) given by the stochastic discount factor process. Furthermore, the Lagrange multiplier  $\lambda$  is given by*

$$\lambda = V_x(0, x_0).$$

**Proof.** The result follows immediately from (3.2) and (3.6). ■

This fact will of course hold also for much more complicated models, including equilibrium models. Recalling this simple result will often simplify the reading of journal articles. As an example, we have the following Corollary, which is trivial, but which looks very deep if one is unaware of the MG approach.

**Corollary 3.3.1** *Let  $V$  be the solution of the HJB equation. We then have*

$$E^P \left[ \int_0^T V_x(t, \hat{X}_t) \cdot \hat{c}_t dt + V_x(T, \hat{X}_T) \cdot \hat{X}_T \right] = V_x(0, x_0)x_0.$$

**Proof.** This is just the budget constraint (3.5), suitable rewritten. ■

### 3.4 The basic PDE in the MG approach

The martingale approach of the previous section looks very nice, but there are, seemingly, some major shortcomings.

- We have no explicit expression for the optimal portfolio weight  $\hat{u}_t$ .
- The formula (3.8), for the optimal consumption is very nice, but it is expressed in the “dual” state variable  $Z = \lambda D$ , rather than as a feedback control in the “primal” state variable  $x$ .
- We would also like to have an explicit expression for the optimal wealth process  $\hat{X}_t$ .

In order to attack these problems, we first note that the multiplier  $\lambda$  is determined by the budget constraint

$$E^Q \left[ \int_0^T e^{-rt} G(t, \lambda M_t) dt + e^{-rT} F(\lambda M_T) \right] = x_0.$$

so from now on we assume that we have computed  $\lambda$ . Furthermore we define the process  $Z$  by

$$Z_t = \lambda M_t, \tag{3.11}$$

so we have

$$\hat{c}_t = G(t, Z_t), \tag{3.12}$$

$$\hat{X}_T = F(Z_T). \tag{3.13}$$

The general strategy is roughly as follows, where for notational simplicity we let  $X_t$  and  $c_t$  denote the optimal value process and the optimal consumption process respectively.



1. From risk neutral valuation is easy to see that  $X_t$  is of the form

$$X_t = H(t, Z_t)$$

where  $H$  satisfies a Kolmogorov backward equation.

2. Using Ito on  $H$  we can compute  $dX$ .
3. We also know that the  $X$  dynamics are of the form

$$dX_t = (\dots) dt + u_t X_t \sigma dW_t. \quad (3.14)$$

4. Comparing these two expressions for  $dX$  we can identify the optimal weight  $u$  from the diffusion part of  $dX$ .
5. We now have  $c$  and  $u$  expressed as functions of the dual variable  $z$ , so we invert the formula  $x = H(t, z)$  to obtain  $z = K(t, x)$ . Substituting this into the formulas for  $u$  and  $c$  will give us  $u$  and  $c$  as feedback controls in the primal state variable  $x$ .
6. Finally, we investigate what the Kolmogorov equation above looks like in the new variable  $x$ .

We now carry out this program and start by noticing that from risk neutral valuation and the formulas (3.12)-(3.13), we have

$$X_t = E^Q \left[ \int_t^T e^{-r(s-t)} G(s, Z_s) ds + e^{-r(T-t)} F(Z_T) \middle| \mathcal{F}_t \right]. \quad (3.15)$$

This allows us to conclude that  $X_t$  can be expressed as

$$X_t = H(t, Z_t)$$

where  $H$ , defined by,

$$H(t, z) = E_{t,z}^Q \left[ \int_t^T e^{-r(s-t)} G(s, Z_s) ds + e^{-r(T-t)} F(Z_T) \right]$$

satisfies a Kolmogorov equation. To find this equation we need, however, to have the  $Q$  dynamics of  $Z$ . Since  $Z = B_t^{-1} L_t$  and the  $L$  dynamics are

$$dL_t = L_t \varphi_t dW_t,$$

we see that the  $P$  dynamics of  $Z$  are

$$dZ_t = -rZ_t dt + Z_t \varphi dW_t$$

where  $\varphi$  is the Girsanov kernel. By the Girsanov Theorem we thus obtain the  $Q$  dynamics as

$$dZ_t = Z_t (\varphi^2 - r) dt + Z_t \varphi dW_t^Q.$$

We now have the following Kolmogorov equation, which is the basic PDE in the martingale approach.

$$\begin{cases} H_t + z(\varphi^2 - r)H_z + \frac{1}{2}\varphi^2 z^2 H_{zz} + c(t, z) - rH & = 0, \\ H(T, z) & = F(z). \end{cases} \quad (3.16)$$

For mnemotechnical purposes we have here used the notation

$$c(t, z) = G(t, z).$$

Having solved the Kolmogorov equation we can use Ito to obtain the  $X$  dynamics as

$$dX_t = (\dots) dt + H_z(t, Z_t) Z_t \varphi dW_t.$$

and, comparing this with (3.14) we obtain the optimal portfolio weight as

$$u(t, z) = \frac{\varphi}{\sigma} \cdot \frac{zH_z(t, z)}{H(t, z)}.$$

We can summarize our findings so far.

**Proposition 3.4.1** *Defining the process  $Z$  by  $Z_t = \lambda M_t$  we have the following formulas for the optimal wealth, consumption, and portfolio weight.*

$$\widehat{X}_t = H(t, Z_t), \quad (3.17)$$

$$\widehat{c}(t, z) = G(t, z), \quad (3.18)$$

$$\widehat{u}(t, z) = \frac{\varphi}{\sigma} \cdot \frac{zH_z(t, z)}{H(t, z)}. \quad (3.19)$$

The function  $H$  is defined by

$$H(t, z) = E_{t,z}^Q \left[ \int_t^T e^{-r(s-t)} \widehat{c}_s ds + e^{-r(T-t)} \widehat{X}_T \right] \quad (3.20)$$

and satisfies the PDE (3.16).

**Remark 3.4.1** *In the application of the MG approach above we have neglected to take into account the constraint that we are not allowed to go short in physical investment, i.e  $u_t \geq 0$ . This constraint implies that the market is not really complete, so formally we are not allowed to use the MG approach. A way out of this predicament is to formally relax the positivity constraint on physical investment. We then have a complete market model and we are free to use the MG approach. If the optimal  $u$  in this extended model is positive then we have obviously also found the optimum for the original model. In a concrete case we thus have to check that  $\widehat{u}$  above is in fact positive.*

### 3.5 An alternative representation of $H$

The function  $H$  in (3.20) is written as an expectation under  $Q$ . Using the stochastic discount factor  $M$  we can of course also write it as the  $P$ -expectation

$$H(t, Z_t) = \frac{1}{M_t} E_{t, Z_t}^P \left[ \int_t^T M_s \hat{c}_s ds + M_T \hat{X}_T \right]$$

or, equivalently, as

$$H(t, z) = \frac{1}{z} E_{t, z}^P \left[ \int_t^T Z_s \hat{c}_s ds + Z_T \hat{X}_T \right]$$

so

$$H(t, z) = \frac{1}{z} H^0(t, z)$$

where  $H^0$  satisfies the PDE

$$\begin{cases} H_t^0 - rzH_z^0 + \frac{1}{2}\varphi^2 z^2 H_{zz}^0 + zG(t, z) & = 0, \\ H^0(T, z) & = zF(z). \end{cases} \quad (3.21)$$

### 3.6 The connection between Kolmogorov and HJB

Looking at Proposition 3.4.1 we see that, in one sense, our control problem is completely solved. We have determined the optimal wealth, portfolio, and consumption, up to the solution of the PDE (3.16). There are, however, two remaining problems to be studied.

- In Proposition 3.4.1 the optimal controls are determined in terms of the variable  $z$ . From an applied point of view it would be much more natural to have the controls expressed as feedback controls in the wealth variable  $x$ .
- The Kolmogorov equation (3.16) is a **linear** PDE for  $H$ , but the HJB equation is **non-linear** for  $V$ . It seems natural to expect that there must be some relation between these equations.

It is, in principle, not hard to obtain expressions for the optimal controls as feedback controls in the  $x$ -variable. We have the relation

$$x = H(t, z),$$

and if this can be inverted in the  $z$  variable, we can express  $z$  as

$$z = K(t, x).$$

We can then simply substitute this into the formulas above to obtain

$$\widehat{c}(t, x) = G(t, K(t, x)), \quad (3.22)$$

$$\widehat{u}(t, x) = \frac{\varphi}{\sigma} \cdot \frac{K(t, x)H_z(t, K(t, x))}{H(t, K(t, x))}. \quad (3.23)$$

It is now natural to investigate what the relevant PDE for  $K(t, x)$  looks like. To do this we recall that by definition we have

$$H(t, K(T, x)) = x,$$

for all  $x$ . Differentiating this identity once in the  $t$  variable and twice in the  $x$  variable gives us, after some reshuffling, the following relations.

$$H_t = -\frac{K_t}{K_x}, \quad H_z = \frac{1}{K_x}, \quad H_{zz} = -\frac{K_{xx}}{K_x^3}.$$

If we plug these relations into the Kolmogorov equation (3.16) we obtain the PDE

$$\begin{cases} K_t + (rx - c)K_x + \frac{1}{2}\varphi^2 K^2 \frac{K_{xx}}{K_x^2} + (r - \varphi^2)K = 0, \\ K(T, x) = \Phi'(x), \end{cases} \quad (3.24)$$

where the boundary condition follows from the relation

$$K(T, F(z)) = z,$$

plus the property that  $F = (\Phi')^{-1}$ .

To understand the nature of this PDE we recall that from Proposition 3.3.1 we have

$$V_x(t, \widehat{X}_t) = Z_t,$$

and since we also have

$$Z_t = K(t, \widehat{X}_t)$$

this implies that we must have the interpretation

$$K(t, x) = V_x(t, x).$$

If we want to double-check this, we can differentiate the HJB equation

$$V_t + U(t, \widehat{c}) + (rx - \widehat{c})V_x - \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0,$$

in the  $x$  variable, while using the optimality condition (3.2) or, equivalently, use the Envelope Theorem on the original HJB equation (3.1). Defining  $K$  by  $K = V_x$  will then again give us (3.24).

We summarize our results as follows.

**Proposition 3.6.1** *With notation as above we have the following results.*

- The process  $Z_t = \lambda M_t$  has the representation

$$Z_t = V_x(t, \widehat{X}_t).$$

- The optimal wealth process is given by

$$\widehat{X}_t = H(t, Z_t),$$

where the function  $H$  is defined by the Kolmogorov equation (3.16).

- The formulas for the optimal portfolio and consumption in Proposition 3.4.1 are mapped into the formulas (3.2)-(3.3) by the change of variable

$$\begin{aligned} x &= H(t, z), \\ z &= K(t, x), \end{aligned}$$

where  $K$  is the functional inverse of  $H$  in the  $z$  variable.

- We have the identification

$$K(t, x) = V_x(t, x).$$

- After the variable change  $z = K(t, x)$ , the Kolmogorov equation (3.16) transforms into the PDE (3.24) for  $K$ . Since  $K = V_x$  this equation is identical to the PDE for  $V_x$  one obtains by differentiating the HJB equation (3.4) w.r.t. the  $x$  variable.

## 3.7 Concluding remarks

From the analysis above we see that there are advantages and disadvantages for the DynP as well as for the MG approach. Schematically, the situation is as follows.

- Using DynP we end up with the highly non linear HJB equation (3.4), which can be very difficult to solve.
- On the positive side for DynP, the controls are expressed directly in the natural state variable  $x$ .
- For the MG approach, the relevant PDE is much easier than the corresponding HJB equation for DynP. This is a big advantage.
- On the negative side for the MH approach, the optimal controls are expressed in the dual variable  $z$  instead of the wealth variable  $x$ , and in order to express the controls in the  $x$  variable, we need to invert the function  $H$  above.

### 3.8 Exercises

**Exercise 3.1** *This exercise is a review of (the scalar version of) the Envelope Theorem. Consider two smooth functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ . For any fixed  $a \in \mathbb{R}$  we study the problem*

$$\max_x f(x, a)$$

*s.t. the constraint*

$$g(x, a) \leq 0.$$

*We are thus viewing  $x$  as the variable and  $a$  as a parameter. We assume that for each  $a$  there is a unique optimal  $x$  which we denote by  $\hat{x}(a)$ , and we assume that  $\hat{x}$  is smooth as a function of  $a$ . We furthermore assume that the constraint is binding for all relevant values of  $a$ , and that the usual “constraint qualifications” are satisfied, so the Lagrangian*

$$L = f(x, a) - \lambda g(x, a)$$

*has a stationary point at  $\hat{x}(a)$ . Now define the optimal value function  $F$  by*

$$F(a) = f(\hat{x}(a), a).$$

(i) *Use the first order conditions on  $L$  and the fact that we have  $g(\hat{x}(a), a) = 0$  for all  $a$ , in order to prove that*

$$\frac{dF}{da}(a) = \frac{\partial f}{\partial a}(\hat{x}(a), a) - \lambda \frac{\partial g}{\partial a}(\hat{x}(a), a)$$

(ii) *Use the result in (i) to verify that in the unconstrained case we have*

$$\frac{dF}{da}(a) = \frac{\partial f}{\partial a}(\hat{x}(a), a)$$

(iii) *Now study a problem of the form*

$$\max_x f(x)$$

*s.t. the constraint*

$$h(x) \leq a.$$

*Use the result in (i) to show that*

$$\frac{dF}{da}(a) = \lambda.$$

**Exercise 3.2** *Use the Envelope Theorem in order to differentiate the HJB equation (3.1) w.r.t. the  $x$  variable. Define  $K$  by  $K = V_x$  and check that  $K$ , thus defined, will indeed satisfy the PDE (3.24).*

## Part II

# Complete Market Equilibrium Models





## Chapter 4

# A Simple Production Model

We now go on to analyze the simplest possible equilibrium model in a production economy, first using DynP and then using the MG approach.

### 4.1 The Model

The model is formally almost exactly the same as in the previous section, but the interpretation is somewhat different. We start with some formal assumptions which are typical for this theory.

**Assumption 4.1.1** *We assume that there exists a constant returns to scale physical production technology process  $S$  with dynamics*

$$dS_t = \alpha S_t dt + S_t \sigma dW_t. \quad (4.1)$$

*The economic agents can invest unlimited positive amounts in this technology, but since it is a matter of physical investment, short positions are not allowed.*

This assumption is perhaps not completely clear, so we need a more precise interpretation, and it runs basically as follows, where, for simplicity, I will refer to the consumption good as “dollars”, but always with the provision that these “dollars” can be invested as well as consumed.

- At any time  $t$  you are allowed to invest dollars in the production process.
- If you, at time  $t_0$ , invest  $q$  dollars, and wait until time  $t_1$  then you will receive the amount of

$$q \cdot \frac{S_{t_1}}{S_{t_0}}$$

in dollars. In particular we see that the return on the investment is linear in  $q$ , hence the term “constant returns to scale”.

- Since this is a matter of physical investment, shortselling is not allowed.

**Remark 4.1.1** A moment of reflections shows that, from a purely formal point of view, investment in the technology  $S$  is in fact **equivalent to the possibility of investing in a risky asset** with price process  $S$ , but again with the constraint that shortselling is not allowed.

We also need a risk free asset, and this is provided by the next assumption.

**Assumption 4.1.2** We assume that there exists a risk free asset in zero net supply with dynamics

$$dB_t = r_t B_t dt,$$

where  $r$  is the short rate process, which will be determined endogenously. The risk free rate  $r$  is assumed to be of the form

$$r_t = r(t, X_t)$$

where  $X$  denotes portfolio value.

Interpreting the production technology  $S$  as above, the wealth dynamics will be given, exactly as before, by

$$dX_t = X_t u_t (\alpha - r) dt + (r_t X_t - c_t) dt + X_t u_t \sigma dW_t.$$

Finally we need an economic agent.

**Assumption 4.1.3** We assume that there exists a representative agent who wishes to maximize the usual expected utility

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right].$$

## 4.2 Equilibrium

We now go on to study equilibrium in our model. Intuitively this is a situation where the agent is optimal and the market clears for the risk free asset, i.e. the optimal weight on the risky investment is 1.

**Definition 4.2.1** An equilibrium of the model is a triple  $\{\hat{c}(t, x), \hat{u}(t, x), r(t, x)\}$  of real valued functions such that the following hold.

1. Given the risk free rate process  $r(t, X_t)$ , the optimal consumption and investment are given by  $\hat{c}$  and  $\hat{u}$  respectively.
2. The market clears for the risk free asset, i.e.

$$\hat{u}(t, x) \equiv 1.$$

In order to determine the equilibrium risk free rate, we now go on to study the optimal consumption/investment problem for the representative agent. A moments reflection will then convince you that, for the utility maximization problem, all formulas in Section 3 are still valid with the modification that  $r$  is replaced by  $r(t, x)$  and discount factors of the form  $e^{-r(T-t)}$  are replaced by

$$\frac{B_t}{B_T} = e^{-\int_t^T r(u, X_u) du}$$

The HJB equation again reads as

$$V_t + U(t, \hat{c}) + (rx - \hat{c})V_x - \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} \cdot \frac{V_x^2}{V_{xx}} = 0, \quad (4.2)$$

where now  $r$  is shorthand for  $r(t, x)$ . The optimal consumption and weight are given by

$$U_c(t, \hat{c}) = V_x(t, x), \quad (4.3)$$

$$\hat{u}(t, x) = -\frac{(\alpha - r)}{\sigma^2} \cdot \frac{V_x(t, x)}{xV_{xx}(t, x)}. \quad (4.4)$$

Using the equilibrium condition  $\hat{u}_t \equiv 1$  we obtain the main result.

**Proposition 4.2.1** *In equilibrium the following hold.*

- The equilibrium short rate is given by  $r(t, \hat{X}_t)$  where

$$r(t, x) = \alpha + \sigma^2 \frac{xV_{xx}(t, x)}{V_x(t, x)}. \quad (4.5)$$

- The dynamics of the equilibrium wealth process are

$$d\hat{X}_t = \left( \alpha \hat{X}_t - \hat{c}_t \right) dt + \hat{X}_t \sigma dW_t. \quad (4.6)$$

- The Girsanov kernel has the form  $\varphi(t, \hat{X}_t)$  where

$$\varphi(t, x) = \frac{r(t, x) - \alpha}{\sigma}, \quad (4.7)$$

or, alternatively,

$$\varphi(t, x) = \sigma \frac{xV_{xx}(t, x)}{V_x(t, x)}. \quad (4.8)$$

- The optimal value function  $V$  is determined by the HJB equation

$$\begin{cases} V_t + U(t, \hat{c}) + (\alpha x - \hat{c})V_x + \frac{1}{2} \sigma^2 x^2 V_{xx} = 0, \\ V(T, x) = \Phi(x). \end{cases} \quad (4.9)$$

where  $\hat{c}$  is determined by (4.3).

Note that although the HJB equation (4.2) for the (non equilibrium) optimal consumption/investment problem is highly non linear, the **equilibrium** PDE (4.9) is drastically simplified and we see that, apart from the  $\hat{c}$  term, it is in fact linear. As we will see in the next section, this is (of course) not a coincidence.

### 4.3 Introducing a central planner

So far we have assumed that the economic setting is that of a **representative agent** investing and consuming in a market, and we have studied the equilibrium for that market.

An alternative to this setup is when, instead of a representative agent, we consider a **central planner**. The difference between these two concepts is that the central planner does not have access to a financial market, and in particular he/she does not have access to the risk free asset  $B$ .

The optimization problem for the central planner is simply that of maximizing expected utility when everything that is not consumed is invested in the production process. This obviously sounds very much like the problem of a representative agent who, in equilibrium, does not invest anything in the risk free asset, so a very natural conjecture is that the equilibrium consumption of the representative agent coincides with the optimal consumption of the central planner. We will see.

The formal problem of the central planner is to maximize

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right].$$

given the wealth dynamics

$$dX_t = (\alpha X_t - c_t) dt + \sigma X_t dW_t.$$

The HJB equation for this problem is

$$\begin{cases} V_t + \sup_c \left\{ U(t, c) + (\alpha x - c)V_x + \frac{1}{2} \sigma^2 x^2 V_{xx} \right\} = 0, \\ V(T, x) = \Phi(x). \end{cases}$$

with the usual first order condition

$$U_c(t, c) = V_x(t, x).$$

Substituting the optimal  $c$  we thus obtain the PDE

$$\begin{cases} V_t + U(t, \hat{c}) + (\alpha x - \hat{c})V_x + \frac{1}{2} \sigma^2 x^2 V_{xx} = 0, \\ V(T, x) = \Phi(x). \end{cases}$$

and we see that this is identical to (4.9). We have thus proved the following result.

**Proposition 4.3.1** *Given assumptions as above, the following hold.*

- *The optimal consumption for the central planner coincides with the equilibrium consumption of the representative agent.*

- *The optimal wealth process for the central planner is identical with the equilibrium wealth process for the representative agent.*

This result implies in particular that the following scheme is valid.

- Solve the (fairly simple) problem for the central planner, thus providing us with the optimal value function  $V$ .
- Define the “shadow interest rate”  $r$  by (4.5).
- Now forget about the central planner and consider the optimal consumption/investment problem of a representative agent with access to the production technology  $S$  and a risk free asset  $B$  with dynamics

$$dB_t = r(t, X_t)B_t dt$$

where  $r$  is defined as above.

- The economy will then be in equilibrium, so  $\hat{u} = 1$ , and we will recover the optimal consumption and wealth processes of the central planner.

## 4.4 The martingale approach

In order to analyze the equilibrium problem using the martingale approach we need to modify our assumption about the short rate ever so slightly. We thus assume that, apart from the production technology  $S$ , the agent can invest in a risk free asset with a short rate process of the form

$$r_t = r(t, Z_t).$$

Note the difference with the earlier assumption  $r_t = r(t, X_t)$ , and see Remark 4.4.2 for comments on no shortselling.

Recalling the results from Proposition 3.4.1 we obtain the optimal wealth, consumption, and portfolio weight as

$$\begin{aligned} \hat{X}_t &= H(t, Z_t), \\ U_c(t, \hat{c}) &= Z_t, \\ \hat{c}(t, z) &= G(t, z), \\ \hat{u}(t, z) &= \frac{\varphi}{\sigma} \cdot \frac{zH_z(t, z)}{H(t, z)}. \end{aligned}$$

where  $H$  has the interpretation

$$H(t, z) = E_{t,z}^Q \left[ \int_t^T B_s^{-1} \hat{c}_s ds + B_T^{-1} \hat{X}_T \right] \quad (4.10)$$

and thus solves the PDE

$$\begin{cases} H_t + z(\varphi^2 - r)H_z + \frac{1}{2}\varphi^2 z^2 H_{zz} + G(t, z) - rH & = 0, \\ H(T, z) & = F(z), \end{cases} \quad (4.11)$$

and where the Girsanov kernel as usual is given by

$$\varphi = \frac{r - \alpha}{\sigma}.$$

The equilibrium condition  $\hat{u} = 1$  gives us the Girsanov kernel  $\varphi$  and the equilibrium rate  $r$  as

$$\varphi(t, z) = \sigma \frac{H(t, z)}{zH_z(t, z)}, \quad (4.12)$$

$$r(t, z) = \alpha + \sigma^2 \frac{H(t, z)}{zH_z(t, z)}. \quad (4.13)$$

This looks nice and easy, but in order to compute  $\varphi$  and  $r$  we must of course solve the PDE (4.11) for  $H$ . On the surface, this PDE looks reasonable nice, but in order to solve it we must of course substitute (4.12)-(4.13) into (4.11). We then have the following result.

**Proposition 4.4.1** *The equilibrium interest rate and the Girsanov kernel are given by*

$$r(t, z) = \alpha + \sigma^2 \frac{H(t, z)}{zH_z(t, z)}. \quad (4.14)$$

$$\varphi(t, z) = \sigma \frac{H(t, z)}{zH_z(t, z)}, \quad (4.15)$$

where  $H$ , defined by,

$$H(t, z) = E_{t,z}^Q \left[ \int_t^T \frac{B_t}{B_s} \hat{c}_s ds + \frac{B_t}{B_T} \hat{X}_T \right] \quad (4.16)$$

solves the PDE

$$\begin{cases} H_t - \alpha z H_z + \frac{1}{2} \sigma^2 \frac{H^2}{H_z^2} H_{zz} + G - (\alpha + \sigma^2) H & = 0, \\ H(T, z) & = F(z). \end{cases} \quad (4.17)$$

**Remark 4.4.1** *Note that the equilibrium condition introduces a nonlinearity into the PDE for the MG approach.*

We may again argue as in Section 3.6, and perform a change of variable by the prescription

$$x = H(t, z) \quad z = K(t, x).$$

Exactly like in Section 3.6 we can then derive the following PDE for  $K$ .

$$K_t + (\alpha + \sigma^2)xK_x - \hat{c}K_x + \frac{1}{2}\sigma^2x^2K_{xx} = 0. \quad (4.18)$$

As in Section 3.6 we also have the identification

$$K(t, x) = V_x(t, x),$$

and equation (4.18) for  $K$  can also be derived by differentiating the PDE (4.9) in the  $x$  variable.

As in Section 3.5 we can give an alternative representation of  $H$ .

**Proposition 4.4.2** *The equilibrium interest rate and the Girsanov kernel are given by*

$$r(t, z) = \alpha + \sigma^2 \frac{H(t, z)}{zH_z(t, z)}. \quad (4.19)$$

$$\varphi(t, z) = \sigma \frac{H(t, z)}{zH_z(t, z)}, \quad (4.20)$$

where the function  $H$  is given by

$$H(t, z) = \frac{1}{z} E_{t,z}^P \left[ \int_t^T Z_s \hat{c}_s ds + Z_T \hat{X}_T \right]$$

so

$$H(t, z) = \frac{1}{z} H^0(t, z)$$

where  $H^0$  satisfies the PDE

$$\begin{cases} H_t^0 - rzH_z^0 + \frac{1}{2}\varphi^2z^2H_{zz}^0 + zG(t, z) = 0, \\ H^0(T, z) = zF(z). \end{cases} \quad (4.21)$$

**Remark 4.4.2** *Fore this type of production model we are facing the problem that if our process  $S$  has the interpretation of physical investment, then we have a shortselling constraint, the market becomes incomplete, and we are not formally allowed to use the MG approach. There seems to exist at least two ways to handle this problem.*

- *We accept the reality of the shortselling constraint and interpret the results above as the equilibrium results in an extended model where shortselling is formally allowed. Since there is in fact no shortselling in equilibrium we then conclude that the extended equilibrium is indeed also an equilibrium for the original model. This, however, leaves open the question whether there can exist an equilibrium in the original model, which is not an equilibrium in the extended model.*

- *We gloss over the problem, abstain from even mentioning it, and hope that it will disappear. This seems to be a rather common strategy in the literature.*

## 4.5 Introducing a central planner

In the DynP approach we introduced, with considerable success, a central planner who maximized expected utility of wealth and consumption

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right].$$

given the wealth dynamics

$$dX_t = (\alpha X_t - c_t) dt + \sigma X_t dW_t.$$

The important assumption here is that the central planner does not have access to the risk free asset  $B$ . This obviously implies that the market is incomplete so, as far as I understand, this implies that we cannot use the usual MG approach. It should be mentioned, however, that Kramkov and Schachermayer has developed a very deep duality theory also for incomplete markets, and it would be interesting to see what one can say about the problem of the central planner using their theory.

## 4.6 Concluding remarks

In Section 3 we found that the Komogorov PDE in the MG approach had a much simpler structure than the HJB equation for the DynP approach. It seems, however, that this advantage of the MG approach over the DynP approach vanishes completely when we move from the pure optimization model to the equilibrium model. Equation (4.17) for  $H$  does not at all look easier than the HJB equation (4.9).

## 4.7 Exercises

**Exercise 4.1** *Consider the case when  $\Phi(x) = 0$ , and*

$$U(t, c) = e^{-\delta t} \ln(c).$$

(a) *Analyze this case using DynP and the Ansatz*

$$V(t, x) = f(t) \ln(x) + g(t),$$

(b) *Analyze the same problem using the martingale method, by applying Proposition 4.4.2.*



**Exercise 4.2** Analyze the case of power utility, i.e. when

$$U(t, c) = e^{-\delta t} \frac{c^{1-\beta}}{1-\beta}$$

for  $\beta > 0$ .

## 4.8 Notes

The model in this section is a simple special case of the general production model of [5].



# Chapter 5

## The CIR Factor Model

We now go on to analyze a simple version of the Cox-Ingersoll-Ross model as described in [5].

### 5.1 The model

In the model some objects are assumed to be given exogenously whereas other objects are determined by equilibrium, and we also have economic agents.

#### 5.1.1 Exogenous objects

We start with the exogenous objects.

**Assumption 5.1.1** *The following objects are considered as given a priori.*

1. A 2-dimensional Wiener process  $W$ .
2. A scalar factor process  $Y$  of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad (5.1)$$

where  $\mu$  is a scalar real valued function and  $\sigma$  is a 2-dimensional row vector function.

3. A constant returns to scale production technology process  $S$  with dynamics

$$dS_t = \alpha(Y_t)S_t dt + S_t \gamma(Y_t) dW_t \quad (5.2)$$

The interpretation of this is that  $Y$  is a process which in some way influences the economy. It could for example describe the weather. The interpretation of the production technology is as in Chapter 4 and we have again a shortselling constraint.

### 5.1.2 Endogenous objects

In this model we also have some processes which are to be determined endogenously in equilibrium. They are as follows, where we use the notation

$$X_t = \text{the portfolio value at time } t,$$

to be more precisely defined below.

1. A risk free asset  $B$ , in zero net supply, with dynamics

$$dB_t = r_t B_t dt$$

where the risk free rate  $r$  is assumed to be of the form

$$r_t = r(t, X_t, Y_t).$$

2. A financial derivative process  $F(t, X_t, Y_t)$ , in zero net supply, defined in terms of  $X$  and  $Y$ , without dividends and with dynamics of the form

$$dF = \beta F dt + F h dW_t$$

where  $\beta$  and  $h$  are of the form

$$\beta = \beta(t, X_t, Y_t), \quad h = h(t, X_t, Y_t),$$

and will be determined in equilibrium.

We also need an important assumption.

**Assumption 5.1.2** *We assume that the  $2 \times 2$  diffusion matrix*

$$\begin{pmatrix} -\gamma & - \\ -h & - \end{pmatrix}$$

*is invertible P-a.s. for all  $t$*

The implication of this assumption is that, apart from the shortselling constraint of  $S$ , the market consisting of  $S$ ,  $F$ , and  $B$  is complete.

### 5.1.3 Economic agents

The basic assumption in [5] is that there are a finite number of agents with identical initial capital, identical beliefs about the world, and identical preferences. In the present complete market setting this implies that we may as well consider a single representative agent. The object of the agent is (loosely) to maximize expected utility of the form

$$E^P \left[ \int_0^T U(t, c_t, Y_t) dt \right]$$

where  $c$  is the consumption rate (measured in dollars per time unit) and  $U$  is the utility function.

## 5.2 The portfolio problem

In this section we discuss the relevant portfolio theory, formulate the agent's optimal control problem and derive the relevant HJB equation.

### 5.2.1 Portfolio dynamics

The agent can invest in  $S$ ,  $F$ , and  $B$  and. We will use the following notation

$$\begin{aligned} X &= \text{portfolio market value,} \\ a &= \text{portfolio weight on } S, \\ b &= \text{portfolio weight on } F, \\ 1 - a - b &= \text{portfolio weight on } B \end{aligned}$$

Using standard theory we see that the portfolio dynamics are given by

$$dX_t = a_t X_t \frac{dS_t}{S_t} + b_t X_t \frac{dF_t}{F_t} + (1 - a_t - b_t) X_t \frac{dB_t}{B_t} - c_t dt$$

where, for simplicity of notation, lower case index  $t$  always indicates running time, but where other variables are suppressed. This gives us the portfolio dynamics as

$$dX_t = X_t \{a(\alpha - r) + b(\beta - r)\} dt + (rX_t - c) dt + X_t \{a\gamma + bh\} dW_t, \quad (5.3)$$

and we write this more compactly as

$$dX_t = X_t m(t, X_t, Y_t, u_t) dt - c_t dt + X_t g(t, X_t, Y_t, u_t) dW_t, \quad (5.4)$$

where we use the shorthand notation

$$u = (a, b),$$

and where  $m$  and  $g$  are defined by

$$m = a[\alpha - r] + b[\beta - r] + r, \quad (5.5)$$

$$g = a\gamma + bh. \quad (5.6)$$

### 5.2.2 The control problem and the HJB equation

The control problem for the agent is to maximize

$$E^P \left[ \int_0^\tau U(t, c_t, Y_t) dt \right]$$

where

$$\tau = \inf \{t \geq 0 : X_t = 0\} \wedge T$$

subject to the portfolio dynamics

$$dX_t = X_t m(t, X_t, Y_t, u_t) dt - c_t dt + X_t g(t, X_t, Y_t, u_t) dW_t,$$

and the control constraints

$$c \geq 0, \quad a \geq 0.$$

The HJB equation for this is straightforward and reads as

$$\begin{cases} V_t + \sup_{c,u} \{U + \mathbf{A}^u V\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0, \end{cases} \quad (5.7)$$

where the infinitesimal operator  $\mathbf{A}^u$  is defined by

$$\mathbf{A}^u V = (xm - c)V_x + \mu V_y + \frac{1}{2}x^2 g^2 V_{xx} + \frac{1}{2}\sigma^2 V_{yy} + xg\sigma V_{xy}.$$

Here, for the vectors  $\sigma$  and  $g$  in  $R^2$ , we have used the notation

$$\begin{aligned} \sigma g &= (\sigma, g), \\ g^2 &= \|g\|^2, \\ \sigma^2 &= \|\sigma\|^2 \end{aligned}$$

where  $(\sigma, g)$  denotes inner product.

### 5.3 Equilibrium

Since  $B$  and  $F$  are in zero net supply, we have the following definition of equilibrium.

**Definition 5.3.1** *An equilibrium is a list of processes*

$$\{r, \beta, h, a, b, c, V\}$$

*such that  $(V, a, b, c)$  solves the HJB equation given  $(r, \beta, h)$ , and the market clearing conditions*

$$a_t = 1, \quad b_t = 0.$$

*are satisfied.*

We will now study the implications of the equilibrium conditions on the short rate  $r$  and the dynamics of  $F$ . We do this by studying the first order conditions for optimality in the HJB equations, with the equilibrium conditions in force.

The first order conditions, with the equilibrium conditions  $a = 1$  and  $b = 0$  inserted, are easily seen to be as follows.

$$(a) \quad x(\alpha - r)V_x + x^2\gamma^2V_{xx} + x\gamma\sigma V_{xy} = 0, \quad (5.8)$$

$$(b) \quad x(\beta - r)V_x + x^2\gamma hV_{xx} + x\sigma hV_{xy} = 0, \quad (5.9)$$

$$(c) \quad U_c = V_x, \quad (5.10)$$

where (a) indicates that it is the FOC for  $a$  etc.

Substituting these conditions into the HJB equation and the portfolio dynamics, will give us the following result.

**Proposition 5.3.1 (The Equilibrium HJB Equation)** *In equilibrium, the following hold.*

- The HJB equation takes the form

$$\left\{ \begin{array}{l} V_t + U(t, \hat{c}) + (\alpha x - \hat{c})V_x + \mu V_y + \frac{1}{2}x^2\gamma^2V_{xx} + \frac{1}{2}\sigma^2V_{yy} + x\sigma\gamma V_{xy} = 0 \\ V(T, x, y) = 0 \\ V(t, 0, y) = 0 \end{array} \right. \quad (5.11)$$

where  $\hat{c}$  is determined by (5.10).

- The equilibrium portfolio dynamics are given by

$$d\hat{X}_t = (\alpha\hat{X}_t - \hat{c}_t)dt + \hat{X}_t\gamma dW_t \quad (5.12)$$

**Remark 5.3.1** *We will see below that “everything” in the model, like the risk free rate, the Girsanov kernel, risk premia etc, is determined by the equilibrium optimal value function  $V$ . It is then important, and perhaps surprising, to note that the equilibrium HJB equation (5.11) is completely determined by **exogenous data**, i.e. by the  $Y$  and  $S$  dynamics. In other words, the equilibrium short rate, risk premia etc, do **not** depend on the particular choice of derivative  $F$  that we use in order to complete the market.*

## 5.4 The short rate and the risk premium for $F$

From the FOC (5.8) for  $a$  we immediately obtain our first main result.

**Proposition 5.4.1** *The equilibrium short rate  $r(t, x, y)$  is given by*

$$r = \alpha + \gamma^2 \frac{xV_{xx}}{V_x} + \gamma\sigma \frac{V_{xy}}{V_x} \quad (5.13)$$

With obvious notation we can write this as

$$r = \alpha - \left(-\frac{xV_{xx}}{V_x}\right) \text{Var}\left(\frac{dX}{X}\right) - \left(-\frac{V_{xy}}{V_x}\right) \text{Cov}\left[\frac{dX}{X}, dY\right]. \quad (5.14)$$

From the equilibrium optimality condition (5.10) for  $b$  we have the following result.

**Proposition 5.4.2** *The risk premium for  $F$  in equilibrium is given by*

$$\beta - r = - \left[ \frac{xV_{xx}}{V_x} \gamma h + \frac{V_{xy}}{V_x} \sigma h \right] \quad (5.15)$$

## 5.5 The martingale measure and the SDF

Since every equilibrium must be arbitrage free, we can in fact push the analysis further. We denote by  $\varphi$  the Girsanov kernel for the likelihood process  $L = \frac{dQ}{dP}$ , so  $L$  has dynamics

$$dL_t = L_t \varphi_t dW_t.$$

We know from arbitrage theory that the martingale conditions for  $S$  and  $F$  are

$$\begin{aligned} r &= \alpha + \gamma \varphi, \\ r &= \beta + h \varphi \end{aligned}$$

On the other hand we have, from (5.13) and (5.15),

$$\begin{aligned} r &= \alpha + \left\{ \frac{xV_{xx}}{V_x} \gamma + \frac{V_{xy}}{V_x} \sigma \right\} \gamma, \\ r &= \beta + \left\{ \frac{xV_{xx}}{V_x} \gamma + \frac{V_{xy}}{V_x} \sigma \right\} h \end{aligned}$$

Using Assumption 5.1.2 we can thus solve for the vector  $\varphi$  to obtain the following important result.

**Proposition 5.5.1** *The Girsanov kernel  $\varphi$  is given by*

$$\varphi = \frac{xV_{xx}}{V_x} \gamma + \frac{V_{xy}}{V_x} \sigma. \quad (5.16)$$

From our earlier experiences with the martingale approach we also expect to have the relation

$$V_x(t, X_t, Y_t) = \lambda M_t,$$

along the equilibrium  $X$ -path, where  $M$  is the stochastic discount factor

$$M_t = B_t^{-1} L_t,$$

and  $\lambda$  is the Lagrange multiplier, which can be written as

$$\lambda = V_x(0, X_0, Y_0).$$



This result is clear from general martingale theory theory, but one can also derive it using a more bare hands approach by first recalling that the dynamics of  $Z_t = \lambda M_t$  are given by

$$dZ_t = -rZ_t dt + Z_t \varphi dW_t,$$

with  $\varphi$  as in (5.16). We can then use the Ito formula on  $V_x$  and the envelope theorem on the HJB equation in equilibrium to compute  $dV_x$ . This is quite messy, but after lengthy calculations we obtain

$$dV_x = -rV_x dt + V_x \varphi dW_t.$$

Comparing this with the  $Z$  dynamics above gives us the following result.

**Proposition 5.5.2** *The stochastic discount factor in equilibrium is given by*

$$M_t = \frac{V_x(t, X_t, Y_t)}{V_x(0, X_0, Y_0)}. \quad (5.17)$$

## 5.6 Risk neutral valuation

We now go on to derive the relevant theory of risk neutral valuation within the model. This can be done in (at least) two ways:

- We can follow the argument in [5] and use PDE techniques.
- We can use more general arbitrage theory using martingale measures.

To illustrate the difference we will in fact present both argument, and we start with the martingale argument. The reader will notice that the modern martingale argument is considerably more streamlined than the traditional PDE argument.

### 5.6.1 The martingale argument

From general arbitrage theory we then immediately have the standard risk neutral valuation formula

$$F(t, x, y) = E_{t,x,y}^Q \left[ e^{-\int_t^T r_s ds} H(X_T, Y_T) \right] \quad (5.18)$$

where  $H$  is the contract function for  $F$ . We have already determined the Girsanov kernel  $\varphi$  by (5.16) so the equilibrium  $Q$ -dynamics of  $X$  and  $Y$  are given by

$$\begin{aligned} d\hat{X}_t &= \hat{X}_t [\alpha + \gamma\varphi] dt - \hat{c}_t dt + \hat{X}_t \gamma dW_t^Q, \\ dY_t &= [\mu + \sigma\varphi] dt + \sigma dW_t^Q. \end{aligned}$$

We thus deduce that the pricing function  $F$  is the solution of the PDE

$$\left\{ \begin{array}{l} F_t + F_x x(\alpha + \gamma\varphi) - cF_x + \frac{1}{2}x^2\gamma^2 F_{xx} \\ + F_y(\mu + \sigma\varphi) + \frac{1}{2}F_{yy}\sigma^2 + xF_{xy}\sigma\gamma - rF = 0 \\ F(T, x, y) = H(x, y) \end{array} \right. \quad (5.19)$$

which is Kolmogorov backward equation for the expectation above.

### 5.6.2 The PDE argument

Using the Ito formula to compute  $dF$  and comparing with the dynamics

$$dF = F\beta dt + Fh dW_t$$

allows us to identify  $\beta$  as

$$\beta = \frac{1}{F} \left\{ F_t + (\alpha x - c)F_x + \mu F_y + \frac{1}{2}x^2\gamma^2 F_{xx} + \frac{1}{2}\sigma^2 F_{yy} + x\sigma\gamma F_{xy} \right\} \quad (5.20)$$

On the other hand we have

$$\beta - r = -\varphi h$$

with  $\varphi$  again given by (5.16), and

$$h = \frac{1}{F} \{ xF_x\gamma + F_y\sigma \} \quad (5.21)$$

so we have

$$\beta = r - \frac{1}{F} \{ xF_x\gamma\varphi + F_y\sigma\varphi \} \quad (5.22)$$

Comparing the expressions (5.20) and (5.22) for  $\beta$  gives us the basic pricing PDE

$$\left\{ \begin{array}{l} F_t + F_x x(\alpha + \gamma\varphi) - cF_x + \frac{1}{2}x^2\gamma^2 F_{xx} \\ + F_y(\mu + \sigma\varphi) + \frac{1}{2}F_{yy}\sigma^2 + xF_{xy}\sigma\gamma - rF = 0 \\ F(T, x, y) = H(x, y) \end{array} \right. \quad (5.23)$$

which is (of course) identical to (5.19). Using Feynman-Kac we then obtain the standard risk neutral valuation formula as

$$F(t, x, y) = E_{t,x,y}^Q \left[ e^{-\int_t^T r_s ds} H(X_T, Y_T) \right] \quad (5.24)$$

## 5.7 Another formula for $\varphi$

We recall the formula

$$\varphi = \frac{xV_{xx}}{V_x}\gamma + \frac{V_{xy}}{V_x}\sigma$$

for the Girsanov kernel. We also recall from the first order condition (5.10) for consumption, that

$$U_c = V_x.$$

Let us now specialize to the case when the utility function has the form

$$U(t, c, y) = e^{-\delta t}U(c) \quad (5.25)$$

Along the equilibrium path we then have

$$V_x = e^{-\delta t}U'(\hat{c})$$

and differentiating this equation proves the following result.

**Proposition 5.7.1** *Under the assumption (5.25), the Girsanov kernel is given by*

$$\varphi = \frac{U''(\hat{c})}{U'(\hat{c})} \{x\hat{c}_x\gamma + \hat{c}_y\sigma\}$$

## 5.8 Introducing a central planner

As in Section 4.3 we now introduce a central planner into the economy. This means that there is no market for  $B$  and  $F$ , so the central planner only chooses the consumption rate, invests everything into  $S$ , and the problem is thus to maximize

$$E^P \left[ \int_0^\tau U(t, c_t, Y_t) dt + \Phi(X_T) \right]$$

subject to the dynamics

$$\begin{aligned} dX_t &= (\alpha X_t - c)dt + X_t\gamma dt, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t \end{aligned}$$

and the constraint  $c \geq 0$ .

The Bellman equation for this problem is

$$\begin{cases} V_t + \sup_c \left\{ U + (\alpha x - c)V_x + \mu V_y + \frac{1}{2}\gamma^2 x^2 V_{xx} + \frac{1}{2}\sigma^2 V_{yy} + x\sigma\gamma V_{xy} \right\} &= 0 \\ V(T, x) &= \Phi(x) \\ V(t, 0) &= 0 \end{cases}$$

We now see that this is exactly the equilibrium Bellman equation (5.11) in the CIR model. We thus have the following result.

**Proposition 5.8.1** *Given assumptions as above, the following hold.*

- *The optimal consumption for the central planner coincides with the equilibrium consumption of the representative agent.*
- *The optimal wealth process for the central planner is identical with the equilibrium wealth process for the representative agent.*

This result implies in particular that the following scheme is valid.

- Solve the (fairly simple) problem for the central planner, thus providing us with the optimal value function  $V$ .
- Define the “shadow interest rate”  $r$  by (5.13), and the Girsanov kernel  $\varphi$  by (5.16).
- For a derivative with contract function  $H$ , define  $F$  by (5.24).
- Define  $h$  and  $\beta$  by (5.21)-(5.22).
- The  $F$  dynamics will now be

$$dF = \beta F dt + F h dW_t.$$

- Now forget about the central planner and consider the optimal consumption/investment problem of a representative agent with access to the production technology  $S$ , the derivative  $F$  and the risk free asset  $B$  with dynamics

$$dB_t = r(t, X_t) B_t dt$$

where  $r$  is defined as above.

- The economy will then be in equilibrium, so  $a = 1$ ,  $b = 0$  and we will recover the optimal consumption and wealth processes of the central planner.

## 5.9 The martingale approach

We now study the equilibrium problem above using martingale techniques. Applying the usual arguments we then want to maximize expected utility

$$E^P \left[ \int_0^\tau U(t, c_t, Y_t) dt + \Phi(X_T) \right]$$

given the budget constraint

$$E^P \left[ \int_0^\tau c_t M_t dt + X_T M_T \right] = x_0$$

where, as usual,  $M$  is the stochastic discount factor defined by

$$M_t = B_t^{-1} L_t,$$

and  $L$  is the likelihood process  $L = dQ/dP$ . We note that  $M$  will be determined endogenously in equilibrium. The Lagrangian for this problem is

$$E^P \left[ \int_0^T \{U - Z_t c_t\} dt + \Phi(X_T) - Z_T X_T \right] + \lambda x_0$$

where

$$Z_t = \lambda M_t.$$

The first order conditions are

$$U_c(t, \hat{c}_t, Y_t) = Z_t, \quad (5.26)$$

$$\Phi'(\hat{X}_T) = Z_T, \quad (5.27)$$

and, comparing (5.26) with (5.10) we have our first result.

**Proposition 5.9.1** *In equilibrium we have the identification*

$$V_x(t, \hat{X}_t, Y_t) = \lambda M_t,$$

where

$$\lambda = V_x(0, x_0, y_0)$$

Denoting the inverse of  $U_c(t, c, y)$  in the  $c$  variable by  $G(t, z, y)$  and the inverse of  $\Phi'$  by  $F$  we can express the optimal consumption and optimal terminal wealth profile as

$$\begin{aligned} \hat{c}(t, z, y) &= G(t, z, y), \\ \hat{X}_T &= F(Z_T). \end{aligned}$$

At this point we need a small assumption in order to obtain a Markovian structure.

**Assumption 5.9.1** *We assume that the equilibrium short rate  $r$  and the equilibrium Girsanov kernel  $\varphi$  have the form*

$$\begin{aligned} r &= r(t, Z_t, Y_t), \\ \varphi &= \varphi(t, Z_t, Y_t). \end{aligned}$$

We can now apply risk neutral valuation to obtain the optimal wealth process  $X$  (for notational convenience we write  $X$  instead of  $\hat{X}$ ).

$$X_t = E^Q \left[ \int_t^T e^{-\int_t^s r_u du} G(s, Z_s, Y_s) ds + e^{-\int_t^T r_u du} F(Z_T) \middle| \mathcal{F}_t \right]$$

and the Markovian structure allows us to express  $X$  as

$$X_t = H(t, Z_t, Y_t)$$

where  $H$  solves a Kolmogorov equation. In order to find this equation we need the  $Q$  dynamics of  $Z$ , and these are easily obtained as

$$dZ_t = (\varphi^2 - r)Z_t dt + Z_t \varphi dW_t^Q.$$

The Kolmogorov equation is now

$$\begin{cases} H_t + \mathcal{A}H + G - rH &= 0, \\ H(T, x, y) &= F(z) \end{cases}$$

where the infinitesimal operator  $\mathcal{A}$  is defined by

$$\mathcal{A}H = (\varphi^2 - r)zH_z + (\mu + \sigma\varphi)H_y + \frac{1}{2}\varphi^2 z^2 H_{zz} + \frac{1}{2}\sigma^2 H_{yy} + \varphi\sigma H_{zy}$$

We now proceed exactly as in Section 4.4 and use Ito to express the  $X$  dynamics as

$$dX_t = (\dots)dt + \{Z_t H_z \varphi + H_y \sigma\} dW_t$$

On the other hand, we know from general theory that the  $X$  dynamics in equilibrium are given by

$$dX_t = (\dots)dt + X_t \gamma dW_t,$$

so, using  $X_t = H(t, Z_t, Y_t)$  we obtain

$$zH_z \varphi + H_y \sigma = H\gamma,$$

giving us

$$\varphi = \frac{H}{zH_z} \gamma - \frac{H_y}{zH_z} \sigma.$$

The martingale condition for  $S$  is obviously

$$r = \alpha + \varphi\gamma,$$

which is our formula for the equilibrium interest rate. We may now summarize.

**Proposition 5.9.2** *The equilibrium interest rate  $r(t, z, y)$  and Girsanov kernel  $\varphi(t, z, y)$  are given by*

$$r = \alpha + \frac{H}{zH_z} \gamma^2 - \frac{H_y}{zH_z} \sigma \gamma, \quad (5.28)$$

$$\varphi = \frac{H}{zH_z} \gamma - \frac{H_y}{zH_z} \sigma. \quad (5.29)$$

Here the function  $H(t, z, y)$  is determined by the PDE

$$\begin{cases} H_t + \mathcal{A}H + G - rH &= 0, \\ H(T, z, y) &= F(z) \end{cases} \quad (5.30)$$

with  $\mathcal{A}$  is defined by

$$\mathcal{A}H = (\varphi^2 - r)zH_z + (\mu + \sigma\varphi)H_y + \frac{1}{2}\varphi^2 z^2 H_{zz} + \frac{1}{2}\sigma^2 H_{yy} + \varphi\sigma H_{zy}$$

and  $r$  and  $\varphi$  replaced by the formulas (5.28)-(5.29).

As in the Section 3.5 we have an alternative representation of  $H$ .

**Proposition 5.9.3** *The function  $H$  can also be written as*

$$H(t, z, y) = \frac{1}{z} H^0(t, z, y)$$

where  $H^0$  is given by

$$H^0(t, z, y) = E_{t,z,y}^P \left[ \int_t^T Z_s \hat{c}_s + Z_T \hat{X}_T \right],$$

and solves the PDE

$$\begin{cases} H_t^0 - rzH_z^0 + \mu H_y^0 + \frac{1}{2} \varphi^2 z^2 H_{zz}^0 + \frac{1}{2} \sigma^2 H_{yy}^0 + \varphi \sigma H_{zy}^0 + zG & = 0, \\ H^0(T, z, y) & = zF(z) \end{cases}$$

**Remark 5.9.1** *Inserting (5.28)-(5.29) into any of the PDE:s above will result in a really horrible PDE, and I am rather at a loss to see how to proceed with that object.*

## 5.10 Exercises

**Exercise 5.1** *Assume that the utility function  $U$  does not depend on  $y$  and has the form*

$$U(t, c) = e^{-\delta t} \frac{1}{1-\beta} c^{1-\beta}$$

where  $\beta$  and  $\delta$  are positive real numbers. We also assume that  $\Phi = 0$ .

(a) *Prove that optimal value function  $V(t, x, y)$  has the form*

$$V(t, x, y) = U(t, x) f(t, y)$$

with  $U$  as above, where  $f$  satisfies the PDE

$$\begin{cases} f_t + [\mu + \sigma\gamma(1-\beta)] f_y + \frac{\sigma^2}{2} f_{yy} + \beta f^{1-\frac{1}{\beta}} + \left[ \alpha(1-\beta) - \delta - \frac{1}{2} \gamma^2 \beta(1-\beta) \right] f & = 0, \\ f(T, y) & = 0 \end{cases}$$

(b) *Compute (in terms of  $f$ ) the equilibrium interest rate  $r$  and the Girsanov kernel  $\varphi$ . Note that they do not depend on the  $x$ -variable.*

**Exercise 5.2** *With  $\Phi = 0$ , assume and that we have log utility, i.e.*

$$U(t, c, y) = e^{-\delta t} \ln(c),$$

(a) Show that the HJB equation has a solution of the form

$$V(t, x, y) = e^{-\delta t} f(t, y) \ln(x) + e^{-\delta t} g(t, y)$$

and derive the relevant PDE:s for  $f$  and  $g$ .

(b) Solve the PDE for  $f$  and derive explicit expressions for  $r$  and  $\varphi$ .

(c) Use the martingale approach and compute the function  $H$  of Section 5.9.

(d) Use  $H$  to derive explicit expressions for  $r$  and  $\varphi$ , and compare with (b).

## 5.11 Notes

This model studied above is a special case of the the model presented in [5], where the authors also allow for several production processes, but where only PDE methods are used. A very general multi-agent equilibrium model, allowing for several production processes, as well as endowments, is studied in detail in [22].



## Chapter 6

# The CIR Interest Rate Model

We now specialize to the model in [6]. In this model the authors study power utility, but all concrete formulas are actually derived under the assumption of log utility, i.e.

$$U(t, c, y) = e^{-\delta t} \ln(c),$$

so we restrict ourselves to this particular case.

### 6.1 Dynamic programming

Given the assumption of log utility, it is easy to see that the HJB equation has a solution of the form

$$V(t, x, y) = e^{-\delta t} f(t, y) \ln(x) + e^{-\delta t} g(t, y)$$

and we obtain the following PDE for  $f$ .

$$\begin{cases} f_t + \mu f_y + \frac{1}{2} \sigma^2 f_{yy} - \delta f + 1 = 0, \\ F(T, y) = 0. \end{cases}$$

Using Feynman-Kac it is easily seen that  $f$  is given by the formula

$$f(t, y) = \frac{1}{\delta} \left[ 1 - e^{-\delta(T-t)} \right].$$

so we have

$$\frac{x V_{xx}}{V_x} = -1, \quad \frac{V_{xy}}{V_x} = 0,$$

and plugging this into the formula (5.13) gives us the short rate as

$$r(t, y) = \alpha(y) - \gamma^2(y).$$

In view of this formula it is now natural to specialize further by assuming that

$$\begin{aligned}\alpha(y) &= \alpha \cdot y, \\ \gamma(y) &= \gamma \cdot \sqrt{y}.\end{aligned}$$

which means that the  $S$  dynamics are of the form

$$dS_t = \alpha S_t Y_t dy + \gamma S_t \sqrt{Y_t} dW_t$$

This gives us

$$r(t, y) = (\alpha - \gamma^2)y,$$

Now, in order to have a positive  $Y$  process, which is necessary for  $\sqrt{Y_t}$  to make sense, we introduce the assumption that the  $Y$  dynamics are of the form

$$dY_t = \{AY_t + B\} dt + \sigma \sqrt{Y_t} dW_t \quad (6.1)$$

where  $A$ ,  $B$  and  $\sigma$  are positive constants so in the earlier notation we have

$$\begin{aligned}\mu(y) &= Ay + B, \\ \sigma(y) &= \sigma \sqrt{y}.\end{aligned}$$

and, using (5.16) for the Girsanov kernel, we have proved the following result.

**Proposition 6.1.1** *For the CIR II model described above the following hold.*

- *The short rate is given by*

$$r(t, Y_t) = (\alpha - \gamma^2)Y_t.$$

- *The short rate dynamics under  $P$  are*

$$dr_t = [A + B_0] dt + \sigma_0 \sqrt{r_t} dW_t,$$

where

$$\begin{aligned}B_0 &= (\alpha - \gamma^2)B, \\ \sigma_0 &= \sqrt{\alpha - \gamma^2} \sigma.\end{aligned}$$

- *The Girsanov kernel is given by*

$$\varphi(t, y) = -\gamma \sqrt{y}.$$

- *The  $Q$  dynamics of  $r$  are*

$$dr_t = [A_0 r_t + B_0] dt + \sigma_0 \sqrt{r_t} dW_t^Q$$

where

$$A_0 = A - \gamma \sigma \sqrt{\alpha - \gamma^2}.$$

**Remark 6.1.1** *The condition guaranteeing that the  $Y$  equation has a positive solution is*

$$2A \geq \sigma^2.$$

*This will also guarantee that the SDE for the short rate has a positive solution. In order to have a positive short rate we obviously also need to assume that*

$$\alpha \geq \gamma^2.$$

## 6.2 Martingale analysis

The problem is to maximize expected utility

$$E^P \left[ \int_0^T e^{-\delta t} \ln(c_t) dt \right]$$

subject to the budget constraint

$$E^P \left[ \int_0^T M_t c_t dt \right] = x_0$$

Performing the usual calculations, we obtain the optimal consumption as

$$\hat{c}_t = \lambda^{-1} M_t^{-1} e^{-\delta t}.$$

From Section 5.9 we recall the function  $H$ , defined by

$$H(t, z, y) = E_{t,z,y}^Q \left[ \int_t^T \frac{B_t}{B_s} \hat{c}_s ds \right].$$

We can also write this as

$$H(t, z, y) = \frac{1}{M_t} E_{t,z,y}^P \left[ \int_t^T M_s \hat{c}_s ds \right].$$

Inserting the expression for  $\hat{c}$  and recalling that  $Z_t = \lambda M_t$  gives us the formula

$$H(t, z, y) = \frac{1}{z} g(t)$$

where

$$g(t) = \frac{1}{\delta} \{ e^{-\delta t} - e^{-\delta T} \}$$

From (5.28) we recall the formula

$$r = \alpha + \frac{H}{zH_z} \gamma^2 - \frac{H_y}{zH_z} \sigma \gamma,$$

so we obtain

$$r(t, y) = \alpha(y) - \gamma^2(y),$$

and we can proceed as in Section 6.1.

## 6.3 Exercises

**Exercise 6.1** *This exercise shows that you can basically generate an arbitrarily chosen process as the short rate in a CIR model with log utility.*

*Consider the CIR setting above with log utility, but with production dynamics of the form*

$$dS_t = S_t Y_t dt + \gamma S_t dW_t,$$

*where  $\gamma$  is a real number and  $Y$  is an arbitrary process. Compute the short rate.*

## 6.4 Notes

The model in this section is the one presented in [6].

## Chapter 7

# Endowment Equilibrium 1: Unit Net Supply

In the previous chapters we have studied equilibrium models in economies with a production technology. An alternative to that setup is to model an economy where each agent is exogenously endowed with a stream of income/consumption. This can be done in several ways, and we start with the simplest one, characterized by unit net supply of risky assets.

### 7.1 The model

In the model some objects are assumed to be given exogenously whereas other objects are determined by equilibrium, and we also have economic agents.

#### 7.1.1 Exogenous objects

We start with the exogenous objects.

**Assumption 7.1.1** *The following objects are considered as given a priori.*

1. A 1-dimensional Wiener process  $W$ .
2. A scalar and strictly positive process  $e$  of the form

$$de_t = a(e_t)dt + b(e_t)dW_t \quad (7.1)$$

where  $a$  and  $b$  is a scalar real valued functions.

The interpretation of this is that  $e$  is an **endowment process** which provides the owner with a consumption stream at the rate  $e_t$  units of the consumption good per unit time, so during the time interval  $[t, t + dt]$  the owner will obtain  $e_t dt$  units of the consumption good.

### 7.1.2 Endogenous objects

The endogenous object in the model are as follows.

1. A risk free asset  $B$ , in zero net supply, with dynamics

$$dB_t = r_t B_t dt$$

where the risk free rate  $r$  is determined in equilibrium.

2. A price dividend pair  $(S, D)$  in unit net supply, where by assumption

$$dD_t = e_t dt.$$

In other words: Holding the asset  $S$  provides the owner with the dividend process  $e$  over the time interval  $[0, T]$ . Since  $S$  is defined in terms of  $e$  we can write the dynamics of  $S$  as

$$dS_t = \alpha_t S_t dt + \gamma_t S_t dW_t$$

where  $\alpha$  and  $\gamma$  will be determined in equilibrium.

3. We stress the fact that, apart for providing the owner with the dividend process  $e$  over  $[0, T]$ , the asset  $S$  gives no further benefits to the owner. In equilibrium we will thus have

$$S_t = \frac{1}{M_t} E^P \left[ \int_t^T M_s e_s ds \middle| \mathcal{F}_t \right],$$

where  $M$  is the equilibrium stochastic discount factor. In particular we will have

$$S_T = 0.$$

### 7.1.3 Economic agents

We consider a single representative agent who wants to maximize expected utility of the form

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

where  $c$  is the consumption rate (measured in dollars per time unit) and  $U$  is the utility function.

**Assumption 7.1.2** *We assume that the agent has initial wealth  $X_0 = S_0$ . In other words: The agent has enough money to buy the right to the dividend process  $Y$ .*

We will use the notation

$$\begin{aligned} u_t &= \text{portfolio weight on the risky asset,} \\ 1 - u_t &= \text{portfolio weight on the risk free asset,} \\ c_t &= \text{rate of consumption.} \end{aligned}$$

### 7.1.4 Equilibrium conditions

The natural equilibrium conditions are that the agent will, at all times, hold the risky asset and that he will consume all dividends. Formally this reads as follows.

$$\begin{aligned} u_t &= 1, && ((S, D) \text{ in unit net supply}), \\ 1 - u_t &= 0, && (B \text{ in zero net supply}), \\ c_t &= e_t, && (\text{market clearing for consumption}). \end{aligned}$$

## 7.2 Dynamic programming

As usual we start by attacking the problem using DynP and, as we will see below, this is not completely trivial. In Section 7.3 we will analyze the same equilibrium problem using the martingale approach, and we will see that, for this particular model, the martingale approach is in fact much more efficient than the dynamic programming method. The reader who wants to go directly to the main results can therefore skip this section and go to Section 7.3.

In order to obtain a Markovian structure we make the following assumption.

**Assumption 7.2.1** *We assume that  $S$ ,  $\alpha$ ,  $\gamma$  and  $r$ , have the following structure, where  $F$  below is a smooth function to be determined in equilibrium.*

$$\begin{aligned} S_t &= F(t, e_t), \\ \alpha_t &= \alpha(T, e_t), \\ \gamma_t &= \gamma(t, e_t), \\ r_t &= r(t, e_t). \end{aligned}$$

It would seem natural to allow the functions above to depend also on the equilibrium wealth process  $X$  but, as we will see, this is not necessary.

### 7.2.1 Formulating the control problem

In equilibrium, the risky asset is in unit net supply so in equilibrium we will have  $X_t = S_t$ , but the individual agent will of course feel no such restriction, and we thus have to determine the dividend rate which will be allocated to the agent given wealth  $x$ , the weight  $u$  on the risky asset, the price  $s$  of the risky asset, and the value  $e$  of the dividend rate process.

This, however, is quite easy. The dollar amount invested in the risky asset is  $ux$  and, given the asset price  $s$ , this implies that the agent is holding  $ux/s$  units of the risky asset, and thus that he obtains a dividend of size

$$\frac{ux}{s} \cdot e \cdot dt$$

over an infinitesimal interval  $[t, t + dt]$ . We emphasize that, here and elsewhere in this chapter, the notation  $s$  is shorthand for  $F(t, e)$ .

The wealth dynamics are thus given by

$$dX_t = uX_t(\alpha_t - r_t)dt + (rX_t - c_t + \frac{ux}{s}e)dt + u_tX_t\gamma_t dW_t, \quad (7.2)$$

and the natural constraints for the agent are

$$c_t \geq 0, \quad (7.3)$$

$$u_t = 0 \quad \text{if } X_t = 0, \quad (7.4)$$

$$c_t \leq e_t \quad \text{if } X_t = 0. \quad (7.5)$$

The first conditions is obvious. The second and third conditions prohibits short-selling of  $S$ , as well as excessive consumption, during periods with zero wealth, thus prohibiting wealth from going negative.

The control problem of the agent is thus to maximize

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

given the  $X$  dynamics (7.2), and the constraints (7.3)-(7.5).

## 7.2.2 The HJB equation

The optimal value function is of the form  $V(t, x, e)$ , and the HJB equation is as follows.

$$\begin{cases} V_t(t, x, e) + \sup_{u, c} \{U(t, c) + \mathbf{A}^{u, c}V(t, x, e)\} = 0 \\ V(T, x, e) = 0 \end{cases}$$

where  $\mathbf{A}^{u, c}$  is given by

$$\mathbf{A}^{u, c}V = ux(\alpha - r)V_x + (rx - c + \frac{ux}{s}e)V_x + \frac{1}{2}u^2x^2\gamma^2V_{xx} + \mu V_e + \frac{1}{2}b^2V_{ee} + ux\gamma bV_{xe},$$

with the constraints (7.3)-(7.5) for  $c$  and  $u$ .

**Remark 7.2.1** *It would perhaps seem natural that, since we have the term  $\frac{ux}{s}e$ , we should study an optimal value function of the form  $V(t, x, e, s)$ . This is, however, not necessary. Since we have assumed that  $S_t = F(t, e_t)$ , the asset price  $S$  will not be a state variable, and the appearance of  $s$  in the HJB equation is, as always, shorthand for  $F(t, e)$ .*

Assuming that the constraints are not binding, the first order conditions are

$$\frac{\partial U}{\partial c}(t, \hat{c}) = V_x(t, x, e),$$

and

$$x(\alpha - r)V_x + \frac{x}{s}eV_x + ux^2\gamma^2V_{xx} + x\gamma bV_{xe} = 0.$$



### 7.2.3 Equilibrium

In order to obtain expressions for the equilibrium risk free rate, and the market price of risk, we would now like to introduce the proper equilibrium conditions into the first order conditions above. The equilibrium concept for the present model is, however, a little bit tricky, and the situation is roughly as follows.

- For our underlying economic model, we have assumed that that  $X_0 = S_0$ .
- This assumptions is tailor made to fit the assumption that  $(S, D)$  is in **unit** net supply.
- Given  $X_0 = S_0$  we expect that, in equilibrium, the agent is at all times holding one unit of  $(S, D)$ , with no money invested in  $B$ , and consuming  $c_t = e_t$ . This will then imply that  $X_t = S_t$ .
- Using DynP however, we have to study the HJB equation for **all** possible combinations of  $x$ ,  $e$ , and  $s$ , and not just for the case  $x = s$ .
- We thus have to extend our equilibrium concept to the case when  $x \neq s$  and introduce this extended concept into the first order conditions above.
- Having done this, we may finally assume that  $X_0 = S_0$ , and study the implications of this assumption.

The extended equilibrium conditions are easily seen to be as follows

$$c(t, x, e) = \frac{x}{F(t, e)} \cdot e, \quad (7.6)$$

$$u(t, x, e) = 1. \quad (7.7)$$

The second condition is obvious. In order to understand the first condition, assume that at time  $t$  you have wealth  $x$ . If you invest all of this into the risky asset, then the dividend rate is  $\frac{x}{s} \cdot e$  and if all this is consumed then your consumption is given by  $c = \frac{x}{s} \cdot e$ . Recalling that  $s = F(t, e)$  gives us (7.7).

**Remark 7.2.2** *In the case when  $x = s$ , the extended equilibrium conditions will obviously coincide with the “naive” equilibrium conditions  $u = 1$  and  $c = e$ . Given an arbitrary initial conditions  $X_0 = x_0$ , we also expect to have*

$$X_t = \frac{x_0}{s_0} S_t$$

for all  $t$ .

Inserting the extended equilibrium conditions into the wealth dynamics gives us the (extended) equilibrium  $X$  dynamics as

$$dX_t = \alpha X_t dt + X_t \gamma dW_t,$$

and from this we conclude, by comparing with the  $S$  dynamics, that

$$X_t = \frac{x_0}{s_0} S_t,$$

or in other words

$$\frac{X_t}{F(t, e_t)} = \frac{x_0}{s_0}. \quad (7.8)$$

Inserting the extended equilibrium conditions into the first order conditions gives us

$$U_c \left( t, \frac{x}{F(t, e)} e \right) = V_x(t, x, e), \quad (7.9)$$

$$r - \frac{e}{s} = \alpha + x \frac{V_{xx}}{V_x} \gamma^2 + \frac{V_{xe}}{V_x} b \gamma. \quad (7.10)$$

From (7.10) and standard arbitrage theory, we can now identify the Girsanov kernel  $\varphi$  as

$$\varphi(t, x, e) = x \frac{V_{xx}}{V_x} \gamma + \frac{V_{xe}}{V_x} b. \quad (7.11)$$

This is more or less expected, but we can in fact obtain a much nicer formula.

**Proposition 7.2.1** *For the original equilibrium, where  $x_0 = s_0$ , the Girsanov kernel process  $\varphi$  is of the form  $\varphi_t = \varphi(t, e_t)$  where the function  $\varphi(t, e)$  is given by*

$$\varphi(t, e) = \frac{U_{cc}(t, e)}{U_c(t, e)} \cdot b(e).$$

**Proof.** From (7.9) we obtain

$$\begin{aligned} V_{xx}(t, x, e) &= \frac{e}{F(t, e)} U_{cc} \cdot \left( t, \frac{xe}{F(t, e)} \right), \\ V_{xe}(t, x, e) &= x \cdot U_{cc} \left( t, \frac{xe}{F(t, e)} \right) \left\{ \frac{F(t, e) - eF(t, e)}{F^2(t, e)} \right\}. \end{aligned}$$

This gives us

$$\varphi(t, x, e) = \frac{xe}{F} \cdot \frac{U_{cc}}{U_c} \gamma + x \frac{U_{cc}}{U_c} \left\{ \frac{F - eF}{F^2} \right\} b. \quad (7.12)$$

From Ito we have

$$\gamma = \frac{F_e}{F} b,$$

and if we plug this, together with the equilibrium conditions  $x = F$  and  $c = e$ , into (7.12) we obtain the stated result. ■

Given our experiences from previous chapters, we expect the stochastic discount factor  $M_t$  to be given by

$$M_t = \frac{V_x(t, X_t, e_t)}{V_x(0, x_0, e_0)}$$

along the equilibrium trajectory, and since  $X_t = S_t = F(t, e_t)$  in equilibrium this would imply that the normalized stochastic discount factor  $Z$  is given by

$$Z_t = V_x(t, F(t, e_t), e_t) = V_x(t, X_t, e_t) = U_c(t, e_t)$$

This can in fact be proved.

**Proposition 7.2.2** *The normalized stochastic discount factor  $Z$  is, along the equilibrium path generated by  $x_0 = s_0$ , given by*

$$Z_t = V_x(t, F(t, e_t), e_t) = V_x(t, X_t, e_t) = U_c(t, e_t).$$

Furthermore, the corresponding equilibrium short rate is given by

$$r(t, e_t) = -\frac{U_{tc}(t, e_t) + a(t, e_t)U_{cc}(t, e_t) + \frac{1}{2}b^2(e_t)U_{ccc}(t, e_t)}{U_c(t, e_t)}$$

**Proof.** Ito's formula gives us

$$\begin{aligned} dV_x(t, X_t, e_t) &= \left\{ V_{tx} + \alpha X_t V_{xx} + \frac{1}{2}\gamma^2 X_t^2 \gamma^2 V_{xxx} + aV_{xe} + \frac{1}{2}b^2 V_{xee} + X_t \gamma b V_{x xe} \right\} dt \\ &+ \{ X_t \gamma V_{xx} + bV_{xe} \} dW_t \end{aligned}$$

From the HJB equation we also have

$$V_t + U\left(t, \frac{xe}{F}\right) + \alpha x V_x + \frac{1}{2}\gamma^2 V_{xx} + aV_e + \frac{1}{2}b^2 V_{ee} + x\gamma b V_{xe} = 0.$$

Differentiating this w.r.t. the  $x$  variable, and plugging the result into the  $V_x$  dynamics above, gives us

$$\begin{aligned} dV_x(t, X_t, e_t) &= -V_x \frac{U_c(t, \frac{X_t e_t}{F}) \frac{e_t}{F} + \alpha V_x + X_t V_{xx} \gamma^2 + \gamma b V_{xe}}{V_x} dt \\ &+ V_x \frac{X_t \gamma V_{xx} + bV_{xe}}{V_x} dW_t \end{aligned}$$

Using (7.11) we can write this as

$$\begin{aligned} dV_x(t, X_t, e_t) &= -V_x \frac{U_c(t, \frac{X_t e_t}{F}) \frac{e_t}{F} + \alpha V_x + X_t V_{xx} \gamma^2 + \gamma b V_{xe}}{V_x} dt \\ &+ V_x \varphi_t dW_t \end{aligned}$$

Using (7.9)-(7.10) and the fact that in equilibrium we have  $F(t, Y_t) = S_t = X_t$ , gives us

$$dV_x = -V_x r_t dt + V_x \varphi_t dW_t,$$

and we can conclude that in equilibrium  $V_x(t, X_t, e_t) = Z_t$ .

Since  $Z_t = U_c(t, e_t)$  we can now use the Ito formula to obtain

$$dZ_t = \left\{ U_{tc} + aU_{cc} + \frac{1}{2}b^2U_{ccc} \right\} dt + U_{cc}\sigma dW_t,$$

and, using Proposition 7.2.1, we can write this as

$$dZ_t = Z_t \frac{U_{tc} + aU_{cc} + \frac{1}{2}b^2U_{ccc}}{U_c} dt + Z_t \varphi dW_t.$$

The short rate can now be identified from the drift term. ■

**Remark 7.2.3** *We note that the short rate and the Girsanov kernel are completely determined by the  $e$ -dynamics and by the utility function, i.e. by the exogenous objects.*

We still have to justify our assumption that  $S$  is of the form  $S_t = F(t, e_t)$  and the similar assumptions about  $\alpha$  and  $\gamma$ . This is in fact easily done. From risk neutral valuation we have

$$S_t = E^Q \left[ \int_t^T e^{-\int_t^s r(u, e_u) du} e_s ds \middle| \mathcal{F}_t \right],$$

so, by the Markovian structure of  $Y$ ,  $S$  is indeed of the form  $S_t = F(t, e_t)$  where  $F$  solves the Kolmogorov backward equation

$$\begin{cases} F_t(t, e) + \{a(e) + \varphi(t, e)b(e)\} F_e(t, e) + \frac{1}{2}b^2(e)F_{ee}(t, e) - r(t, e)F(t, e) + e = 0, \\ F(T, e) = 0 \end{cases}.$$

By applying the Ito formula to  $F$ , we now obtain  $\alpha$  and  $\gamma$  as

$$\begin{aligned} \alpha(t, e) &= \frac{1}{F(t, e)} \left[ F_t(t, e) + a(e)F_e(t, e) + \frac{1}{2}b^2(e)F_{ee}(t, e) \right], \\ \gamma(t, e) &= \frac{b(e)F_e(t, e)}{F(t, e)}. \end{aligned}$$

## 7.3 The martingale approach

We now go on to study the model above using the martingale approach, and this turns out to be much easier than using DynP.

### 7.3.1 The control problem

We assume again that the initial wealth of the agent is given by  $X_0 = S_0$ . The agent's control problem is then to maximize

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

subject to the following constraints.

$$\begin{aligned} c_t &\geq 0, \\ E^P \left[ \int_0^T M_t c_t dt \right] &\leq S_0. \end{aligned}$$

As usual,  $M$  denotes the stochastic discount factor. The first constraint is obvious and the second one is the budget constraint.

Since the asset  $S$  provides the owner with the income stream defined by  $e$  and nothing else (see Section 7.1.2) we can apply arbitrage theory to deduce that

$$S_0 = E^P \left[ \int_0^T M_t e_t dt \right].$$

We can thus rewrite the budget constraint as

$$E^P \left[ \int_0^T M_t c_t dt \right] \leq E^P \left[ \int_0^T M_t e_t dt \right].$$

This optimization problem is, however, identical to the one in Section 8.3, so the Lagrangian is again given by

$$E^P \left[ \int_0^T \{U(t, c_t) - \lambda M_t c_t dt\} \right] + \lambda E^P \left[ \int_0^T M_t e_t dt \right],$$

where  $\lambda$  is the Lagrange multiplier. and the optimality condition for  $c$  is thus

$$U_c(t, c_t) = Z_t, \tag{7.13}$$

where

$$Z_t = \lambda M_t.$$

### 7.3.2 Equilibrium

As in Section 8.3.2 we make the natural assumption that the processes  $\alpha$ ,  $\gamma$  and  $r$  are of the form

$$\begin{aligned} \alpha_t &= \alpha(t, Z_t, e_t), \\ \gamma_t &= \gamma(t, Z_t, e_t), \\ r_t &= r(t, Z_t, e_t). \end{aligned}$$

The equilibrium conditions are

$$\begin{aligned} u_t &\equiv 1, & (S \text{ in unit net supply}), \\ 1 - u_t &\equiv 0, & (B \text{ in zero net supply}), \\ c_t &\equiv Y_t, & (\text{market clearing for consumption}). \end{aligned}$$

It is now surprisingly easy to derive formulas for the equilibrium short rate and the equilibrium Girsanov kernel. The clearing condition  $c = y$  and the optimality condition (7.13) gives us

$$Z_t = U_c(t, e_t),$$

so we have

$$dZ_t = \left\{ U_{ct}(t, e_t) + a(e_t)U_{cc}(t, e_t) + \frac{1}{2}b^2(e_t)U_{ccc}(t, e_t) \right\} dt + b(e_t)U_{cc}(t, e_t)dW_t.$$

Using the formula

$$dZ_t = -r_t Z_t dt + Z_t \varphi_t dW_t.$$

we can thus identify the equilibrium rate and the Girsanov kernel as follows.

**Proposition 7.3.1** *The equilibrium short rate is given by*

$$r(t, e) = - \frac{U_{ct}(t, e) + a(e)U_{cc}(t, e) + \frac{1}{2}b^2(e)U_{ccc}(t, e)}{U_c(t, e)} \quad (7.14)$$

and we see that the short rate  $r$  does in fact not depend explicitly on  $z$ . Furthermore, the Girsanov kernel is given by

$$\varphi(t, e) = \frac{U_{cc}(t, e)}{U_c(t, e)} \cdot b(e). \quad (7.15)$$

We have thus re-derived the results that we obtained earlier by the dynamic programming approach, and we note that by using the martingale approach we have a drastic simplification of the computational effort.

**Remark 7.3.1** *In the proof above it seems that we have only used the consumption market clearing condition*

$$c_t \equiv e_t,$$

and not at all the clearing conditions for the risky and the risk free assets

$$\begin{aligned} 1 - u_t &\equiv 0, \\ u_t &\equiv 1. \end{aligned}$$

It is, however, easy to see that the clearing condition  $c = e$  actually implies the other two. The equilibrium consumption stream  $c = e$  discussed above, can certainly be replicated by holding exactly one unit of  $(S, D)$  and putting zero weight on the risk free asset. It now follows from market completeness and the martingale representation theorem that this is the unique portfolio providing the holder with the consumption stream  $e$ .

### 7.3.3 Log utility

To exemplify we now specialize to the log utility case when the local utility function is of the form

$$U(t, c) = e^{-\delta t} \ln(c).$$

In this case we have

$$U_c = \frac{1}{c} e^{-\delta t}, \quad U_{tc} = -\frac{\delta}{c} e^{-\delta t}, \quad U_{cc} = -\frac{1}{c^2} e^{-\delta t}, \quad U_{ccc} = \frac{2}{c^3} e^{-\delta t}$$

Plugging this into the formula (7.14) gives us the short rate as

$$r(t, e) = \delta + \frac{a(e)}{e} - \frac{b^2(e)}{e^2}.$$

Given this expression it is natural to specialize further to the case when the  $e$  dynamics are of the form

$$de_t = ae_t dt + be_t dW_t,$$

where (with a slight abuse of notation)  $a$  and  $b$  in the right hand side are real constants, so that

$$a(e) = a \cdot e, \quad b(e) = b \cdot e.$$

We then obtain a constant short rate of the form

$$r = \delta + a - b^2.$$

## 7.4 Extending the model

In the previous sections we have assumed that the endowment process  $Y$  satisfies an SDE of the form

$$de_t = a(e_t)dt + b(e_t)dW_t.$$

A natural extension of this setup would of course be to consider a factor model of the form

$$\begin{aligned} de_t &= a(e_t, Y_t)dt + b(e_t, Y_t)dW_t, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t \end{aligned}$$

where  $Y$  is an underlying factor process, and  $W$  is a two-dimensional Wiener process. In this section we will extend our endowment theory to include a fairly general model for the endowment process, and as a special case we will consider the factor model above.

### 7.4.1 The general scalar case

We extend the earlier model by simply assuming that the (scalar) endowment process has the structure

$$de_t = a_t dt + b_t dW_t, \quad (7.16)$$

where  $W$  is a  $k$ -dimensional Wiener process, and where the scalar process  $a$  and the  $k$ -dimensional row vector process  $b$  are adapted to some given filtration  $\mathbf{F}$  (which, apart from  $W$ , may include many other driving processes). This setup will obviously include various factor models, and will also include non-Markovian models. We assume that we have  $N + 1$  random sources in the model.

In order to build the model we introduce as usual the asset-dividend pair  $(S, D)$  where

$$dD_t = e_t dt,$$

and we assume, as before, that  $S$  is in unit net supply. The interpretation is again that  $S$  provides the holder with the endowment  $e$  (and nothing else). We then introduce a risk free asset  $B$  and a number of derivatives  $F_1, \dots, F_N$  which are defined in terms of the random sources, so that the market consisting of  $S, B, F_1, \dots, F_N$  is complete.

We can now apply the usual martingale approach, and a moment of reflection will convince you that the argument in Section 7.3.2 goes through without any essential change. We thus conclude that for this extended model we have the following result.

**Proposition 7.4.1** *If the endowment process  $e$  has dynamics according to (7.16), then the following hold.*

- The equilibrium short rate process is given by

$$r_t = - \frac{U_{ct}(t, e) + a_t U_{cc}(t, e_t) + \frac{1}{2} \|b_t\|^2 U_{ccc}(t, e_t)}{U_c(t, e_t)}. \quad (7.17)$$

- The Girsanov kernel is given by

$$\varphi_t = \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot b_t. \quad (7.18)$$

**Remark 7.4.1** *We note that the measure transformation from  $P$  to  $Q$  only affects the Wiener process  $W$  driving the endowment process  $e$ . The distribution of other driving processes will thus **not** be changed.*

### 7.4.2 The multidimensional case

A natural extension of the model in Section 7.4.1 would be to consider, not only one scalar endowment process, but a finite number of endowment processes  $e_1, \dots, e_d$ , with dynamics

$$de_{it} = a_{it} dt + b_{it} dW_t, \quad i = 1, \dots, d, \quad (7.19)$$



where  $W$  is  $k$ -dimensional Wiener (with  $k \geq d$ ) and we have  $N + 1$  random sources in the filtration. We then introduce the price dividend pairs  $(S_1, D_1), \dots, (S_d, D_d)$  where

$$dD_{it} = e_{it}dt.$$

As usual we assume that the risky  $S_1, \dots, S_d$  assets are in unit net supply and that asset  $i$  gives the owner the right to the dividend process  $D_i$ . We also assume the existence of a risk free asset  $B$ , and we assume the existence of a sufficient number of derivative assets in zero net supply, such that the market is complete.

This model looks, *prima facie*, more general than the model of Section 7.4.1, but this is in fact not the case.

Assuming a representative agent with utility  $U(t, c)$  and denoting the aggregate endowment by  $\eta$  so

$$\eta_t = \sum_{i=1}^d e_{it} \quad (7.20)$$

we see that the optimization problem of the representative agent is to maximize expected utility

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

subject to the (aggregate) budget constraint

$$E^P \left[ \int_0^T M_t c_t dt \right] \leq E^P \left[ \int_0^T M_t \eta_t dt \right]$$

The equilibrium market clearing condition is of course

$$\hat{c}_t = \eta_t.$$

From a formal point of view this is exactly the same problem that we studied above apart from the fact that  $e$  is replaced by  $\eta$ . We may thus copy the result from Proposition 7.4.1 to state the following result.

**Proposition 7.4.2** *Write the aggregate endowment process  $\eta$  dynamics as*

$$d\eta_t = a_t dt + b_t dW_t,$$

where

$$a_t = \sum_{i=1}^d a_{it}, \quad b_t = \sum_{i=1}^d b_{it},$$

then the following hold.

- The equilibrium short rate process is given by

$$r_t = - \frac{U_{ct}(t, \eta_t) + a_t U_{cc}(t, \eta_t) + \frac{1}{2} \|b_t\|^2 U_{ccc}(t, \eta_t)}{U_c(t, \eta_t)}. \quad (7.21)$$

- The Girsanov kernel is given by

$$\varphi_t = \frac{U_{cc}(t, \eta_t)}{U_c(t, \eta_t)} \cdot b_t. \quad (7.22)$$

We finish this section by noting that although the case of a multi-dimensional endowment process will formally reduce to the case of a scalar endowment, the multidimensional case may still lead to new computational and structural problems.

Suppose, for example, that we model each  $e_i$  as a Markov process of the form

$$de_{it} = a_i(e_{it})dt + b_i(e_{it})dW_t, \quad i = 1, \dots, d,$$

where  $a_i$  is a deterministic real valued function, and  $b_i$  is a deterministic  $k$ -dimensional row vector function. Then the aggregate endowment will have dynamics

$$d\eta_t = \left\{ \sum_{i=1}^d a_i(e_{it}) \right\} dt + \left\{ \sum_{i=1}^d b_i(e_{it}) \right\} dW_t,$$

so  $\eta$  is **not** Markov. In particular, if each  $e_i$  is GBM, this does **not** imply that  $\eta$  is GBM.

### 7.4.3 A factor model

We exemplify the theory of the previous section by considering a factor model of the form

$$de_t = a(e_t, Y_t)dt + b(e_t, Y_t)dW_t, \quad (7.23)$$

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t. \quad (7.24)$$

where  $W$  is a standard two dimensional Wiener process. For simplicity we assume log utility, so

$$U(t, c) = e^{-\delta t} \ln(c).$$

In this case the equilibrium rate and the Girsanov kernel will be of the form  $r_t = r(e_t, Y_t)$ ,  $\varphi_t = \varphi(e_t, Y_t)$  and after some easy calculations we obtain

$$\begin{aligned} r(e, y) &= \delta + \frac{a(e, y)}{e} - \frac{\|b(e, y)\|^2}{e^2}, \\ \varphi(e, y) &= -\frac{b(e, y)}{e}. \end{aligned}$$

Given these expressions it is natural to make the further assumption that  $a$  and  $b$  are of the form

$$\begin{aligned} a(e, y) &= e \cdot a(y), \\ b(e, y) &= e \cdot b(y), \end{aligned}$$

which implies

$$\begin{aligned} r(y) &= \delta + a(y) - \|b(y)\|^2 \\ \varphi(y) &= -b(y). \end{aligned}$$

We now specialize further to the case when

$$\begin{aligned} a(y) &= a \cdot y, \\ b(y) &= \sqrt{y} \cdot b, \end{aligned}$$

and in order to guarantee positivity of  $Y$  we assume

$$\begin{aligned} \mu(y) &= \beta + \mu \cdot y, \\ \sigma(y) &= \sigma \cdot \sqrt{y} \end{aligned}$$

where  $2\beta \geq \|\sigma\|^2$ . We then have the following result.

**Proposition 7.4.3** *Assume that the model has the structure*

$$\begin{aligned} de_t &= ae_t Y_t dt + e_t b \sqrt{Y_t} dW_t, \\ dY_t &= \{\beta + \mu Y_t\} dt + \sigma \sqrt{Y_t} dW_t. \end{aligned}$$

*Then the equilibrium short rate and the Girsanov kernel are given by*

$$\begin{aligned} r_t &= \delta + (a - \|b\|^2) Y_t, \\ \varphi_t &= \sqrt{Y_t} \cdot b. \end{aligned}$$

We thus see that we have essentially re-derived the Cox-Ingersoll-Ross short rate model, but now within an endowment framework.

We finish this section with a remark on the structure of the Girsanov transformation. Let us assume that, for a general utility function  $U(t, c)$ , the processes  $e$  and  $Y$  are driven by independent Wiener processes, so the model has the form

$$\begin{aligned} de_t &= a(e_t, Y_t) dt + b(e_t, Y_t) dW_t^e, \\ dY_t &= \mu(Y_t) dt + \sigma(Y_t) dW_t^Y. \end{aligned}$$

where  $W^e$  and  $W^Y$  are independent and where  $b$  and  $\sigma$  are scalar. Then the Girsanov kernel has the vector form

$$\varphi_t = \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot [b(e_t, Y_t), 0]$$

so the likelihood dynamics are

$$dL_t = L_t \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot b(e_t, Y_t) dW_t^e,$$

implying that the Girsanov transformation will only affect  $W^e$  and not  $W^Y$ .

## 7.5 Exercises

**Exercise 7.1** Consider a model with log utility

$$U(t, c) = e^{-\delta t} \ln(c)$$

and scalar endowment following GBM

$$de_t = ae_t dt + be_t W_t,$$

where  $a$  and  $b$  are real numbers and  $W$  is  $P$ -Wiener.

(a) Compute the  $P$  and  $Q$  dynamics of  $S$ .

(b) Use the result of (a) and let  $T \rightarrow \infty$ . What do you get?

**Exercise 7.2** Consider a model with power utility

$$U(t, c) = e^{-\delta t} \frac{1}{\gamma} c^\gamma,$$

where  $\gamma < 1$ , and scalar GBM endowment as in the previous exercise. Compute the short rate and the Girsanov kernel. Compute the  $S$ -dynamics when  $T \rightarrow \infty$ .

**Exercise 7.3** Consider a model with exponential utility

$$U(t, c) = e^{-\delta t} \frac{1}{\gamma} e^{-\gamma c}.$$

and scalar endowment with dynamics

$$de_t = (b - ae_t)dt + \sigma\sqrt{e_t}dW_t$$

where  $a$ ,  $b$  and  $\sigma$  are positive real numbers such that  $2b > \sigma^2$  (so that  $e$  stays positive). Compute the short rate, the Girsanov kernel, and the  $S$ -dynamics.

## 7.6 Notes

Basic references for endowment models are [20] and [22]. See also [11] and [24]. In the literature, endowment models are typically analyzed using martingale methods. The DynP treatment above seems to be new but, having seen it, you quickly realize why the martingale approach is preferable.

## Chapter 8

# Endowment Equilibrium 2: Zero Net Supply

In this chapter we will study a small variation of the endowment model discussed in the previous chapter, the difference being that the risky assets are now assumed to be in zero net supply. The zero net supply model is a little bit more messy to analyze than the unit net supply model of the previous chapter, and it leads to exactly the same results, so the the present chapter can be regarded as optional reading.

### 8.1 The model

The model is almost identical to the one in Chapter 7, apart from the fact that the definition of risky asset is slightly modified.

#### 8.1.1 Exogenous objects

We start with the exogenous objects.

**Assumption 8.1.1** *The following objects are considered as given a priori.*

1. A 1-dimensional Wiener process  $W$ .
2. A scalar and strictly positive process  $e$  of the form

$$de_t = a(e_t)dt + b(e_t)dW_t \quad (8.1)$$

where  $a$  and  $b$  is a scalar real valued functions.

The interpretation of this is that  $e$  is a an endowment process which provides the owner with a consumption stream at the rate  $e_t$  units of the consumption good per unit time. In other words: During an infinitesimal interval  $[t, t + dt]$ , the agent receives  $e_t \cdot dt$  units of the consumption good. So far this is exactly like in the previous chapter. The novelty, as we will see below, is in the interpretation of the risky asset.

### 8.1.2 Endogenous objects

In this model we also have some processes which are to be determined endogenously in equilibrium. They are as follows, where we use the notation

$$X_t = \text{the portfolio value at time } t,$$

to be more precisely defined below.

1. A risk free asset  $B$ , in zero net supply, with dynamics

$$dB_t = r_t B_t dt$$

where the risk free rate  $r$  is determined in equilibrium.

2. A financial derivative process  $S_t$ , in zero net supply, defined in terms of  $e$ , without dividends. More precisely we assume that  $S$  is the price of a European derivative which, at exercise date  $T$ , will give the owner the (positive) amount  $\Psi(e_T)$  units of the consumption good. We write the dynamics of  $S$  as

$$dS_t = \alpha_t S_t dt + \gamma_t S_t dW_t$$

where  $\alpha$  and  $\gamma$  will be determined in equilibrium.

We see that the difference between this model and the model in Chapter 7 is that the risky asset is in zero net supply in the present model, while it was in unit net supply in the model in Chapter 7. The assumption that  $S$  is the price of a contingent claim of the form  $\Phi(e_T)$  is not important.  $S$  could in fact also have been defined as the price of the consumption stream  $e$ , thus making the present model almost identical to the one in Chapter 7. The difference would again be the difference between zero and unit net supply.

### 8.1.3 Economic agents

We consider a single representative agent, and the object of the agent is to maximize expected utility of the form

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

where  $c$  is the consumption rate (measured in dollars per time unit) and  $U$  is the utility function. We assume that the agent has no initial wealth so  $X_0 = 0$ .

The difference between the setting of present chapter and that of Chapter 7 is as follows.

- In Chapter 7 the agent had no initial endowment, but she had sufficient initial wealth to buy the right to the endowment process  $e$ .
- In the present chapter, the agent is exogenously provided with the right to the endowment process  $e$  but, apart from this, she has no initial wealth.

The difference between the two models thus seems to be very small and one is left with the feeling that there is perhaps only a linguistic difference between the models. As we will see, the equilibrium results are in fact identical for both models, so the reader may choose to skip the present chapter entirely.

As with the previous model, the martingale approach is much more efficient than the dynamic programming approach and, in fact, the dynamic programming approach turns out to be extremely difficult to implement.

## 8.2 Dynamic programming

We now go on to study the portfolio problem for the agent and the induced equilibrium price system. The formulation of the portfolio problem is, however, not trivial and requires a detailed discussion.

**Remark 8.2.1** *In this section on the DynP approach there is a considerable amount of hand waving and rather loose arguments, as well as wishful thinking. The section on the martingale approach is, however, OK.*

### 8.2.1 Formulating the control problem

The portfolio dynamics of the agent are given by

$$dX_t = u_t(\alpha_t - r_t)dt + (rX_t + Y_t - c_t)dt + u_t\gamma_t dW_t, \quad (8.2)$$

where  $u$  is the dollar amount invested in the risky asset  $S$ , which implies that the amount  $X_t - u_t$  is invested in the risk free asset  $B$ . An obvious, but naive, way of formulating the portfolio problem of the agent is now to maximize expected utility

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

given the portfolio dynamics (8.2), no constraint on the risky investment  $u_t$ , and the obvious consumption constraint

$$c_t \geq 0.$$

This, however, leads to a nonsensical problem, since there is nothing to stop the agent from consuming an infinite amount and thus becoming infinitely happy. The consequence of this hedonistic behavior is of course that wealth will become negative, but since we have no positivity constraint on  $X$  and no penalty term of the form  $E^P [\Phi(X_T)]$  in the objective function, we cannot prohibit the agent from consuming an arbitrarily large amount at each point in time.

This is an old predicament which we have encountered earlier, and then we solved it in a rather elegant way by introducing the stopping time  $\tau$  defined by

$$\tau = \inf \{t \geq 0 : X_t = 0\}$$

and we reformulated the objective criterion so as to maximize

$$E^P \left[ \int_0^{\tau} U(t, c_t) dt \right].$$

In other words: When you run out of money the game is over.

This worked well for the earlier models without endowment, but for the present model with a positive endowment process, there is no good reason why the game should be over when the wealth hits the value zero, since we can consume at the endowment rate, i.e. put  $c_t = e_t$  and still keep the wealth non-negative.

It would of course seem natural to introduce the state constraint

$$X_t \geq 0, \quad \text{for all } t,$$

but, as we have stated earlier, dynamic programming does not allow for state constraints, so this approach cannot be taken. Instead we need to reformulate the problem in such a way that we implicitly force the wealth process to stay non negative. We can in fact achieve this by introducing the following constraints.

- We allow no shortselling when  $X_t = 0$ . This holds for of the risky as well as for the risk free asset.
- When  $X_t > 0$  we allow any positive consumption rate, but when  $X_t = 0$  we introduce the constraint  $c_t \leq e_t$ .

We may thus finally formulate the control problem of the agent as that of maximizing

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

subject to the dynamics

$$\begin{aligned} dX_t &= u_t(\alpha_t - r_t)dt + (rX_t + Y_t - c_t)dt + u_t\gamma_t dW_t, \\ X_0 &= 0, \end{aligned}$$

and the constraints

$$u_t = 0, \quad \text{if } X_t = 0 \tag{8.3}$$

$$c_t \geq 0, \quad \text{for all } t, \tag{8.4}$$

$$c_t \leq e_t, \quad \text{if } X_t = 0. \tag{8.5}$$

In order to have a Markovian structure we also need to assume the following.

**Assumption 8.2.1** *The processes  $\alpha$ ,  $\gamma$  and  $r$  are of the form*

$$\alpha_t = \alpha(t, X_t, e_t),$$

$$\gamma_t = \gamma(t, X_t, e_t),$$

$$r_t = r(t, X_t, e_t).$$



### 8.2.2 The HJB equation

For the problem discussed above, the optimal value function is of the form  $V(t, x, e)$ , and the HJB equation is as follows.

$$\begin{cases} V_t(t, x, e) + \sup_{u, c} \{U(t, c) + \mathbf{A}^{u, c} V(t, x, e)\} = 0 \\ V(T, x, e) = 0 \end{cases}$$

where  $\mathbf{A}^{u, c}$  is given by

$$\mathbf{A}^{u, c} V = u(\alpha - r)V_x + (rx + y - c)V_x + \frac{1}{2}u^2\gamma^2 V_{xx} + aV_e + \frac{1}{2}b^2 V_{ee} + u\gamma b V_{xe},$$

and where  $c$  and  $u$  are under the constraints (8.3) -(8.5).

Assuming (somewhat optimistically) that the constraints are not binding, the first order conditions are

$$\frac{\partial U}{\partial c}(t, \hat{c}) = V_x(t, x, e), \quad (8.6)$$

$$\hat{u} = \frac{r - \alpha}{\gamma^2} \left( \frac{V_x}{V_{xx}} \right) - \frac{b}{\gamma} \left( \frac{V_{xe}}{V_{xx}} \right). \quad (8.7)$$

### 8.2.3 Equilibrium

The natural equilibrium conditions are as follows.

$$\begin{aligned} u_t &\equiv 0, & (S \text{ in zero net supply}), \\ X_t - u_t &\equiv 0, & (B \text{ in zero net supply}), \\ c_t &\equiv e_t, & (\text{market clearing for consumption}). \end{aligned}$$

From the first two conditions we get

$$X_t \equiv 0,$$

so the equilibrium is characterized by the fact that the optimal wealth is identically equal to zero, and all endowments are immediately consumed. In particular, the condition  $X_t = 0$  implies that  $\alpha$ ,  $\gamma$  and  $r$  only depends on  $t$  and  $y$ .

If we insert the equilibrium condition  $u = 0$  into the first order condition for  $u$  we obtain the following formula for the equilibrium interest rate

$$r(t, e) = \alpha(t, y) + \left( \frac{V_{xe}(t, 0, e)}{V_x(t, 0, e)} \right) b(e)\gamma(t, e).$$

From this we see that the Girsanov kernel  $\varphi$  for the transition from  $P$  to the martingale measure  $Q$  is given by

$$\varphi_t = \left( \frac{V_{xe}(t, 0, e_t)}{V_x(t, 0, e_t)} \right) b(e_t).$$

We can in fact obtain a much nicer formula for the the Girsanov kernel  $\varphi$ . From (8.6), the market clearing condition  $c_t = e_t$ , and the fact that in equilibrium we have  $X_t = 0$ , we obtain

$$U_c(t, e) = V_x(t, 0, e), \quad U_{cc}(t, e) = V_{xe}(t, 0, e).$$

We have thus proved the first item of the following result, which is identical to the corresponding result in the previous chapter. We leave the proof of the second item to the interested (and brave) reader.

**Proposition 8.2.1** *The Girsanov kernel is given by*

$$\varphi_t = \frac{U_{cc}(t, e_t)}{U_c(t, Y_t)} b(e_t).$$

*The equilibrium short rate is given by the formula*

$$r(t, e) = - \frac{U_{ct}(t, e) + a(e)U_{cc}(t, e) + \frac{1}{2}b^2(e)U_{ccc}(t, e)}{U_c(t, e)}$$

In order to determine  $\alpha$  and  $\gamma$  we apply risk neutral valuation to obtain

$$S_t = E^Q \left[ e^{-\int_t^T r_s ds} \Psi(e_T) \middle| \mathcal{F}_t \right],$$

so  $S_t = F(t, e_t)$  where  $F$  solves the Kolmogorov backward equation

$$\begin{cases} F_t(t, e) + \{a(e) + \varphi(t, y)b(e)\} F_e(t, e) + \frac{1}{2}b^2(e)F_{ee}(t, e) - r(t, e)F(t, e) = 0, \\ F(T, e) = \Phi(e) \end{cases}.$$

By applying the Ito formula to  $F$ , we now obtain  $\alpha$  and  $\gamma$  as

$$\begin{aligned} \alpha(t, e) &= \frac{1}{F(t, e)} \left( F_t(t, e) + a(e)F_y(t, e) + \frac{1}{2}b^2(e)F_{ee}(t, e) \right), \\ \gamma(t, e) &= \frac{b(e)F_e(t, e)}{F(t, e)}. \end{aligned}$$

## 8.3 The martingale approach

We now go on to study the model above using the martingale approach, and the big advantage of this approach, compared to dynamic programming, is that there is no problem to introduce a non negativity constraint on  $X$ .

### 8.3.1 The control problem

The agent's control problem now consists in maximizing

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

subject to the dynamics

$$\begin{aligned} dX_t &= u_t(\alpha_t - r_t)dt + (rX_t + e_t - c_t)dt + u_t\gamma_t dW_t, \\ X_0 &= 0. \end{aligned}$$

The natural constraint is, apart from the usual positivity constraint on consumption, some sort of nonnegativity constraint on wealth to rule out unlimited consumption. This goal can, however, easily be achieved by simply introducing the constraint that terminal wealth should equal zero. We thus have the constraints

$$\begin{aligned} c_t &\geq 0, \\ X_T &= 0. \end{aligned}$$

We thus see that the control problem is equivalent to the problem of maximizing

$$E^P \left[ \int_0^T U(t, c_t) dt \right]$$

over the class of adapted non-negative consumption processes  $c$ , satisfying the budget constraint

$$E^P \left[ \int_0^T M_t c_t dt \right] \leq E^P \left[ \int_0^T M_t e_t dt \right],$$

where the stochastic discount factor  $M$  is defined as usual by

$$M_t = B_t^{-1} L_t,$$

and where  $L$  is the likelihood process for the transition from  $P$  to  $Q$ , so that

$$dL_t = \varphi_t L_t dW_t,$$

where

$$\varphi_t = \frac{\alpha_t - r_t}{\gamma_t}.$$

The Lagrangian for the programming problem above is given by

$$E^P \left[ \int_0^T \{U(t, c_t) - \lambda M_t c_t\} dt \right] + \lambda E^P \left[ \int_0^T M_t e_t dt \right],$$

where  $\lambda$  is the Lagrange multiplier. The optimality condition for  $c$  is thus

$$U_c(t, c_t) = Z_t, \tag{8.8}$$

where

$$Z_t = \lambda M_t.$$

### 8.3.2 Equilibrium

In order to study equilibrium, we now make the following natural assumption.

**Assumption 8.3.1** *The processes  $\alpha$ ,  $\gamma$  and  $r$  are of the form*

$$\begin{aligned}\alpha_t &= \alpha(t, Z_t, e_t), \\ \gamma_t &= \gamma(t, Z_t, e_t), \\ r_t &= r(t, Z_t, e_t).\end{aligned}$$

The equilibrium conditions are the same as before, namely

$$\begin{aligned}u_t &\equiv 0, & (S \text{ in zero net supply}), \\ X_t - u_t &\equiv 0, & (B \text{ in zero net supply}), \\ c_t &\equiv e_t, & (\text{market clearing for consumption}).\end{aligned}$$

It is now surprisingly easy to derive an expression for the equilibrium short rate. From the clearing condition  $c = y$  and the optimality condition (8.8) we obtain

$$Z_t = U_c(t, e_t),$$

so we have

$$dZ_t = \left\{ U_{ct}(t, e_t) + a(e_t)U_{cc}(t, e_t) + \frac{1}{2}b^2(e_t)U_{ccc}(t, e_t) \right\} dt + b(e_t)U_{cc}(t, e_t)dW_t.$$

On the other hand, we know from general theory that the dynamics of  $Z$  are

$$dZ_t = -r_t Z_t dt + Z_t \varphi_t dW_t.$$

We can thus identify the equilibrium rate and the Girsanov kernel as follows.

**Proposition 8.3.1** *The equilibrium short rate is given by*

$$r(t, e) = -\frac{U_{ct}(t, e) + a(e)U_{cc}(t, e) + \frac{1}{2}b^2(e)U_{ccc}(t, e)}{U_c(t, e)} \quad (8.9)$$

and we see that the short rate  $r$  does in fact not depend explicitly on  $z$ . Furthermore, the Girsanov kernel is given by

$$\varphi(t, e) = \frac{b(e)U_{cc}(t, e)}{U_c(t, e)}. \quad (8.10)$$

**Remark 8.3.1** *We see that the formulas for the short rate and the Girsanov kernel for the present model, with zero net supply for  $S$ , are identical to the corresponding formulas in the model of Chapter 7, with unit net supply for  $S$ . This is expected, given the fact that in equilibrium the agent will, in both models, consume  $Y_t$  per unit time for all  $t$ .*

## 8.4 Notes

Basic references for endowment models are [20] and [22]. See also [11] and [24]. In the literature, endowment models are typically analyzed using martingale methods.

## Chapter 9

# The Existence of a Representative Agent

In the previous equilibrium models we have (bravely) assumed the existence of a representative agent. As the reader will have noticed, this assumption greatly facilitates the analysis, since the equilibrium conditions become very easy to handle. The object of the present chapter is to show that, for a fairly general class of models containing multiple agents with heterogeneous preferences, we can actually **prove** that there exist a representative agent.

### 9.1 The model

For simplicity choose to study a multi-agent version of the model in Chapter 7. We could also study a much more general model, but then the main (and very simple) idea would be harder so see.

#### 9.1.1 Exogenous objects

We start with the exogenous objects.

**Assumption 9.1.1** *The following objects are considered as given a priori.*

1. An  $n$ -dimensional Wiener process  $W$ .
2. An  $n$ -dimensional strictly positive (in all components) column vector process  $e = (e_1, \dots, e_n)'$  with dynamics of the form

$$de_t = a_t dt + b_t dW_t \quad (9.1)$$

where  $a$  is an (adapted)  $R^n$  valued process and  $b$  is an adapted process taking values in the space of  $n \times n$  matrices. With obvious notation we will write the dynamics of  $e_i$  as

$$de_{it} = a_{it} dt + b_{it} dW_t \quad (9.2)$$

The interpretation of this is again that, for each  $i$ ,  $e_i$  is an endowment process which provides the owner with a consumption stream at the rate  $e_{it}$  units of the consumption good per unit time.

### 9.1.2 Endogenous objects

The endogenous objects in the model are as follows.

1. A risk free asset  $B$ , in zero net supply, with dynamics

$$dB_t = r_t B_t dt$$

where the risk free rate  $r$  is determined in equilibrium.

2. A sequence of price dividend pairs  $\{(S^i, D^i) : i = 1, \dots, n\}$ , all in unit net supply, where by assumption

$$dD_t^i = e_{it} dt.$$

In other words: Holding the asset  $S^i$  provides the owner with the dividend rate  $e_i$ . We write the dynamics of  $S^i$  as

$$dS_t^i = \alpha_{it} S_t^i dt + \gamma_{it} S_t^i dW_t, \quad i = 1, \dots, n.$$

where  $\alpha$  and  $\gamma$  will be determined in equilibrium.

### 9.1.3 Economic agents

We consider  $d$  economic agents who want to maximize expected utility of the form

$$E^P \left[ \int_0^T U_i(t, c_{it}) dt \right], \quad i = 1, \dots, d,$$

where  $c_i$  is the consumption rate and  $U_i$  is the utility function for agent  $i$ . We assume that  $U_i$  is strictly concave in the  $c$  variable, and we also need an assumption on initial wealth.

**Assumption 9.1.2** Denoting the wealth process of agent  $i$  by  $X_i$  we assume that

$$\sum_{i=1}^d X_{i0} = \sum_{j=1}^n S_0^j$$

In other words: As a group, the agents have enough money to buy the dividend paying assets  $S^1, \dots, S^n$ .

We will use the notation

$$\begin{aligned} u_{ijt} &= \text{portfolio weight for agent } i \text{ on the risky asset } S^j, \\ u_{it} &= (u_{i1t}, \dots, u_{int}), \quad \text{portfolio weights process for the risky assets} \\ 1 - \sum_{j=1}^n u_{ijt} &= \text{portfolio weight for agent } i \text{ on the risk free asset,} \\ c_{it} &= \text{consumption rate for agent } i. \end{aligned}$$

### 9.1.4 Equilibrium definition

The natural equilibrium conditions are

- The aggregate net demand will, at all times, be exactly one unit of each asset  $S^1, \dots, S^n$ .
- There is zero net demand of the risk free asset  $B$ .
- The consumption market will clear.

Formally this reads as follows.

**Definition 9.1.1** *An equilibrium is a family of portfolio vector processes  $\{u_{it}^*\}_{i=1}^d$ , consumption rates  $\{c_{it}^*\}_{i=1}^d$ , and asset price processes  $(S_t^1, \dots, S_t^n)$  such that*

1. *Given the asset prices  $(S_t^1, \dots, S_t^n)$ , the portfolio vector process  $u_{it}^*$  and the consumption rate process  $c_{it}^*$  are optimal for agent  $i$ .*
2. *The markets for risky assets clear:*

$$\sum_{i=1}^d u_{ijt} X_{it} = S_t^j, \quad j = 1, \dots, n.$$

3. *There is zero net demand for the risk free asset:*

$$\sum_{i=1}^n X_{it} \left( 1 - \sum_{j=1}^n u_{ijt} \right) = 0.$$

4. *The consumption market clears:*

$$\sum_{i=1}^d c_{it} = \sum_{j=1}^n e_{jt}.$$

## 9.2 The optimization problem of the individual agent

In this chapter we do not prove existence of equilibrium, so we need the following assumption.

**Assumption 9.2.1** *We assume the existence of an equilibrium, with a corresponding stochastic discount factor process  $D^*$ .*

Using the martingale approach, the problem of the agent  $i$  is that of maximizing

$$E^P \left[ \int_0^T U_i(t, c_{it}) dt \right],$$

subject to the budget constraint

$$E \left[ \int_0^T M_t^* c_{it} dt \right] \leq x_{i0}.$$

The Lagrange function for this is

$$\int_0^T \{U_i(t, c_{it}) - \lambda_i^* M_t^* c_{it}\} dt + \lambda_i^* x_{i0}.$$

where  $\lambda_i^*$  is the Lagrange multiplier for agent  $i$ . Assuming an interior optimum, this gives us the first order condition

$$U'_{ic}(t, c_{it}^*) = \lambda_i^* M_t^*,$$

where  $\lambda_i^*$  is determined by

$$E \left[ \int_0^T M_t^* c_{it}^* dt \right] = x_{i0}.$$

We thus see that the equilibrium is characterized by the following conditions.

$$U'_{ic}(t, c_{it}^*) = \lambda_i^* M_t^*, \quad (9.3)$$

$$E \left[ \int_0^T M_t^* c_{it}^* dt \right] = x_{i0}, \quad (9.4)$$

$$\sum_{i=1}^d c_{it}^* = \eta_t, \quad (9.5)$$

where the aggregate endowment  $\eta$  is given by

$$\eta_t = \sum_{j=1}^n e_{jt}. \quad (9.6)$$

### 9.3 Constructing the representative agent

Let us consider the equilibrium of the previous chapter, with corresponding equilibrium stochastic discount factor  $D^*$ , consumption rates  $c_1^*, \dots, c_d^*$ , and Lagrange multipliers  $\lambda_1^*, \dots, \lambda_d^*$ . These objects will, in particular, satisfy the conditions (9.3)-(9.5).

We now define the utility function  $U$  for the representative agent as follows.

**Definition 9.3.1** *The utility function  $u : R_+ \times R_+ \rightarrow R$  is defined by*

$$U(t, c) = \sup_{c_1, \dots, c_d} \sum_{i=1}^d \frac{1}{\lambda_i^*} U_i(t, c_i)$$



subject to the constraints

$$\begin{aligned} \sum_{i=1}^d c_i &= c, \\ c_i &\geq 0, \quad i = 1, \dots, d. \end{aligned}$$

For a given  $c$  we denote the optimal  $c_1, \dots, c_d$  by  $\hat{c}_1(c), \dots, \hat{c}_d(c)$ .

Using elementary optimization theory, we know that (for a given  $c \in R_+$ ) there exists a nonnegative Lagrange multiplier  $q(c)$  such that the Lagrange function

$$\sum_{i=1}^d \frac{1}{\lambda_i^*} U_i(t, c_i) - q(c) \left\{ \sum_{i=1}^d c_i - c \right\},$$

is maximized by  $\hat{c}_1(c), \dots, \hat{c}_d(c)$ . Assuming an interior optimum, we thus see that  $\hat{c}_1(c), \dots, \hat{c}_d(c)$  are characterized by the first order conditions

$$U'_{ic}(t, \hat{c}_i(c)) = \lambda_i^* q(c), \quad i = 1, \dots, d. \quad (9.7)$$

From the Envelope Theorem we also know that

$$U'_c(t, c) = q(c). \quad (9.8)$$

## 9.4 The existence result

Let us again consider the multi-agent market model given above, with the corresponding equilibrium, characterized by the price system  $(S^*, B^*)$ , the stochastic discount factor  $D^*$ , consumption policies  $c_1^*, \dots, c_d^*$ , and the corresponding Lagrange multipliers  $\lambda_1^*, \dots, \lambda_d^*$ . Now let us consider the same market but with a single agent, namely the representative agent of the previous section, with the utility function specified in Definition 9.3.1, and initial wealth  $x_0 = \sum_{i=1}^d x_{i0}$ . Using the martingale approach, the problem of the representative agent is that of maximizing

$$E^P \left[ \int_0^T U(t, c_t) dt \right],$$

subject to the budget constraint

$$E \left[ \int_0^T M_t^* c_t dt \right] \leq x_0.$$

The Lagrange function for this is

$$\int_0^T \{U(t, c_t) - \lambda M_t^* c_t\} dt + \lambda x_0.$$

where  $\lambda$  is the Lagrange multiplier for the representative agent. Assuming an interior optimum, this gives us the first order condition

$$U'_c(t, \hat{c}_t) = \lambda M_t^*, \quad (9.9)$$

where  $\lambda$  is determined by

$$E \left[ \int_0^T M_t^* \hat{c}_t dt \right] = x_0. \quad (9.10)$$

We can now formulate the main result of this chapter.

**Theorem 9.4.1 (i)** *The equilibrium price system  $(S^*, B^*)$ , and stochastic discount factor  $D^*$  is also an equilibrium for the single agent defined by Definition 9.3.1, so*

$$\hat{c}_t = \eta_t,$$

where  $\eta$  is defined in (9.6)

(ii) *In equilibrium, the multiplier  $\lambda$  for the representative agent is given by*

$$\lambda = 1.$$

(iii) *The multi agent equilibrium consumption processes  $c_{1t}^*, \dots, c_{dt}^*$  are given by*

$$c_{it}^* = \hat{c}_i(\eta_t),$$

where  $\hat{c}_i$  is given by Definition 9.3.1 and  $\eta_t = \sum_{j=1}^n e_{jt}$ .

**Proof.** Since the optimization problem for the representative agent is convex, the optimal consumption process  $\hat{c}$  and the Lagrange multiplier  $\lambda$  are uniquely determined by the conditions (9.9)-(9.10). It is thus enough to show that

$$U'_c(t, \eta_t) = M_t^*, \quad (9.11)$$

$$E \left[ \int_0^T M_t^* \eta_t dt \right] = x_0, \quad (9.12)$$

$$c_{it}^* = \hat{c}_i(\eta_t). \quad (9.13)$$

From (9.7) we have

$$U'_{ic}(t, \hat{c}_i(\eta_t)) = \lambda_i^* q(\eta_t), \quad i = 1, \dots, d.$$

and from the equilibrium condition (9.3) we also have

$$U'_{ic}(t, c_{it}^*) = \lambda_i^* M_t^*, \quad i = 1, \dots, d.$$

Since  $U'_{ic}(t, c)$  is strictly decreasing in  $c$ , and since

$$\sum_{i=1}^d c_{it}^* = \sum_{i=1}^d \hat{c}_i(\eta_t) = \eta_t,$$

it is easy to deduce that we have

$$\begin{aligned} q(\eta_t) &= M_t^*, \\ \hat{c}_i(\eta_t) &= c_{it}^*, \quad i = 1, \dots, d, \end{aligned}$$

and we have thus proved (9.13).

From (9.8) we have

$$U'_c(t, \eta_t) = q(\eta_t),$$

and since  $q(\eta_t) = M_t^*$  we obtain

$$U'_c(t, \eta_t) = M_t^*,$$

which proves (9.11).

The relation (9.13) follows from (9.4)-(9.5) and the relation  $x_0 = \sum_i^d x_{i0}$ . ■

This result shows that every multi-agent equilibrium can be realized by studying the (much simpler) equilibrium problem for the corresponding representative agent. Note however, that in order to construct the representative agent, we need to know the equilibrium Lagrange multipliers  $\lambda_1^*, \dots, \lambda_1^*$  for the multi-agent model.

This finishes the abstract theory, but in the next chapter we will study two concrete examples to see how the theory can be applied.



# Chapter 10

## Two examples with multiple agents

In this chapter we exemplify the theory developed in the previous chapter by studying two concrete examples.

### 10.1 Log utility with different subsistence levels

In this example we consider a market with  $d$  agents, where the individual utility functions are of the form

$$U_i(t, c) = e^{-\delta t} U_i(c), \quad i = 1, \dots, d.$$

where

$$U_i(c) = \ln(c - \bar{c}_i), \quad i = 1, \dots, d.$$

The interpretation is that  $\bar{c}_i$  is the lowest acceptable level of consumption for agent  $i$ . We also assume that there exists a scalar endowment process  $e$  with dynamics

$$de_t = a_t dt + b_t dW_t$$

where  $a$  and  $b$  are adapted and where  $W$  can be multi dimensional. Agent  $i$  has an initial wealth which is a fraction  $\beta_i$  of the total value (at time  $t = 0$ ) of the endowment, where obviously

$$\sum_{j=1}^d \beta_j = 1.$$

In order to determine the utility function of the representative agent we must solve the following optimization problem where we have used the notation  $\alpha_i = \lambda_i^{-1}$ .

$$\max_{c_1, \dots, c_d} \sum_{i=1}^d \alpha_i \ln(c_i - \bar{c}_i)$$

under the constraints

$$\sum_{i=1}^d c_i = c,$$

and the positivity constraints

$$c_i > \bar{c}_i, \quad i = 1, \dots, d.$$

The Lagrangian of the problem is

$$L = \sum_{i=1}^d \alpha_i \ln(c_i - \bar{c}_i) - q \sum_{i=1}^d c_i + qc$$

with first order conditions

$$\frac{\alpha_i}{c_i - \bar{c}_i} = q, \quad i = 1, \dots, d.$$

This gives us

$$\hat{c}_i = \bar{c}_i + \frac{\alpha_i}{q}.$$

We can now determine  $q$  from the constraint  $\sum_{i=1}^d \hat{c}_i = c$ . We obtain

$$\sum_{i=1}^d \bar{c}_i + \frac{1}{q} \sum_{j=1}^d \alpha_j = c,$$

and, introducing the notation

$$\bar{c} = \sum_{i=1}^d \bar{c}_i$$

for aggregate minimum consumption level, we have

$$q = \frac{\sum_{j=1}^d \alpha_j}{c - \bar{c}}.$$

This gives us the optimal individual consumption as

$$\hat{c}_i = \bar{c}_i + \alpha_i \frac{c - \bar{c}}{\sum_{j=1}^d \alpha_j}, \quad i = 1, \dots, d, \quad (10.1)$$

so the utility function for the representative agent is

$$U(t, c) = e^{-\delta t} U(c),$$

where

$$U(c) = \left( \sum_{j=1}^d \alpha_j \right) \ln(c - \bar{c}) + \sum_{i=1}^d \alpha_i \ln \alpha_i - \left( \sum_{j=1}^d \alpha_j \right) \ln \left( \sum_{j=1}^d \alpha_j \right).$$

This gives us

$$U'(c) = \frac{\sum_1^d \alpha_j}{c - \bar{c}},$$

which is no surprise, since we know from the Envelope Theorem, that  $q = U'(c)$ .

From Theorem 9.4.1 we know that we have

$$Z_t = M_t = U_c(t, e_t).$$

so we obtain

$$M_t = e^{-\delta t} \frac{\sum_1^d \alpha_j}{e_t - \bar{c}}.$$

In particular we have  $M_0 = 1$  so in equilibrium we have

$$\sum_1^d \alpha_j = e_0 - \bar{c}.$$

We thus see that the equilibrium stochastic discount factor is given by

$$M_t = e^{-\delta t} \frac{e_0 - \bar{c}}{e_t - \bar{c}}.$$

and from (10.1) we obtain the equilibrium consumption as

$$\hat{c}_{it} = \bar{c}_i + \alpha_i \frac{e_t - \bar{c}}{e_0 - \bar{c}}$$

It now remains to determine  $\alpha_1, \dots, \alpha_d$  and to this end we use the budget constraint

$$E^P \left[ \int_0^T M_t \hat{c}_{it} dt \right] = \beta_i E^P \left[ \int_0^T M_t e_t dt \right], \quad i = 1, \dots, d.$$

We obtain, after some calculations,

$$\alpha_i = \frac{\delta}{1 - e^{-\delta T}} E^P \left[ \int_0^T M_t \{\beta_i e_t - \bar{c}_i\} dt \right], \quad i = 1, \dots, d.$$

Using Proposition 7.4.1 we then have the following result.

**Proposition 10.1.1** *With notation as above, the following hold.*

1. *The equilibrium consumption is given by*

$$\hat{c}_{it} = \bar{c}_i + \frac{(e_t - \bar{c})}{e_0 - \bar{c}} \frac{\delta}{1 - e^{-\delta T}} E^P \left[ \int_0^T M_t \{\beta_i e_t - \bar{c}_i\} dt \right], \quad i = 1, \dots, d,$$

2. *The stochastic discount factor is given by*

$$M_t = e^{-\delta t} \frac{e_0 - \bar{c}}{e_t - \bar{c}}.$$

3. The equilibrium short rate is given by

$$r_t = \delta - \frac{a_t}{e_t - \bar{c}} - \frac{1}{2} \frac{\|b_t\|^2}{(e_t - \bar{c})^2}.$$

4. The Girsanov kernel is given by

$$\varphi_t = -\frac{b_t}{e_t - \bar{c}}.$$

## 10.2 Log and square root utility

In this example we consider a market with only two agents, where the individual utility functions are of the form

$$\begin{aligned} U_1(t, c) &= e^{-\delta t} \ln(c), \\ U_2(c) &= 2e^{-\delta t} \sqrt{c}. \end{aligned}$$

The endowment process  $e$  has dynamics

$$de_e = a_t dt + b_t dW_t,$$

and the initial wealth of agent  $i$  is a fraction  $\beta_i$  of the total value of the endowment stream  $e$ .

The utility function for the representative agent will obviously have the form  $U(t, c) = e^{-\delta t} U(c)$ , where  $U$  is the optimal value function for the following optimization problem where we have used the notation  $\alpha_i = \lambda_i^{-1}$ ,  $i = 1, 2$ .

$$\max_{c_1, c_2} \alpha_1 \ln(c_1) + 2\alpha_2 \sqrt{c_2}$$

subject to the constraint

$$c_1 + c_2 = c.$$

The Lagrangian of this is

$$L = \alpha_1 \ln(c_1) + 2\alpha_2 \sqrt{c_2} - q(c_1 + c_2) + qc$$

and we obtain

$$\hat{c}_1 = \frac{\alpha_1}{q}, \quad \hat{c}_2 = \left( \frac{\alpha_2}{q} \right)^2. \quad (10.2)$$

Plugging this into the constraint  $c_1 + c_2 = c$  gives us a second order equation for  $q$ , and after some calculations we obtain

$$q = \frac{\alpha_1}{2c} \left\{ 1 + \sqrt{1 + \gamma c} \right\} \quad (10.3)$$

where we have used the notation

$$\gamma = \left( \frac{\alpha_2}{\alpha_1} \right)^2.$$



From the Envelope Theorem we thus have

$$U_c(t, c) = e^{-\delta t} \frac{\alpha_1}{2c} \left\{ 1 + \sqrt{1 + \gamma c} \right\}$$

and from general theory we know that

$$Z_t = \lambda M_t = U_c(t, e_t).$$

In order to have  $M_t = U_c(t, e_t)$ , we use the fact that  $M_0 = 1$ , and normalize by choosing  $\alpha_1$  and  $\alpha_2$  such that  $U_c(0, e_0) = 1$ . This gives us

$$\alpha_1 = \frac{2e_0}{1 + \sqrt{1 + \gamma e_0}}. \quad (10.4)$$

The stochastic discount factor  $M$  is thus given by

$$M_t = e^{-\delta t} \frac{e_0}{e_t} \frac{1 + \sqrt{1 + \gamma e_t}}{1 + \sqrt{1 + \gamma e_0}}.$$

We can now compute the  $\gamma$  by using the budget constraint for agent 1, i.e.

$$E^P \left[ \int_0^T M_t \widehat{c}_{1t} dt \right] = \beta_1 E^P \left[ \int_0^T M_t e_t dt \right].$$

From (10.2)-(10.3) we have

$$\widehat{c}_{1t} = \frac{2e_t}{1 + \sqrt{1 + \gamma e_t}}$$

so the budget constraint takes the form

$$\frac{2e_0}{\delta} \frac{1 - e^{-\delta T}}{1 + \sqrt{1 + \gamma e_0}} = \frac{\beta_1 e_0}{1 + \sqrt{1 + \gamma e_0}} E^P \left[ \int_0^T e^{-\delta t} \left( 1 + \sqrt{1 + \gamma e_t} \right) dt \right]$$

giving us the equation

$$\frac{2}{\delta} (1 - e^{-\delta T}) = \beta_1 E^P \left[ \int_0^T e^{-\delta t} \left( 1 + \sqrt{1 + \gamma e_t} \right) dt \right]. \quad (10.5)$$

This equation determines  $\gamma$ , and (10.4) will then determine  $\alpha_1$ . We summarize as follows.

**Proposition 10.2.1** *With  $\gamma$  determined by (10.5), the following hold.*

1. *The equilibrium consumption plans are given by*

$$\begin{aligned} \widehat{c}_{1t} &= \frac{2e_t}{1 + \sqrt{1 + \gamma e_t}}, \\ \widehat{c}_{2t} &= \frac{4\gamma e_t^2}{(1 + \sqrt{1 + \gamma e_t})^2} \end{aligned}$$

2. *The stochastic discount factor is given by*

$$M_t = e^{-\delta t} \frac{e_0}{e_t} \frac{1 + \sqrt{1 + \gamma e_t}}{1 + \sqrt{1 + \gamma e_0}}.$$



## Part III

# Models with Partial Information



# Chapter 11

## Stating the Problem

In the previous chapters we have silently assumed that all processes, price processes as well as underlying factor processes, are **perfectly observable**, but we will now relax this assumption. To take a concrete example, let us recall the simple CIR model from Chapter 5. We have an underlying factor process  $Y$  with dynamics

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t,$$

and we have a constant returns to scale production technology process  $S$  with dynamics

$$dS_t = \alpha(Y_t)S_t dt + S_t \gamma(Y_t) dW_t.$$

Previously we also assumed the existence of a representative agent with utility of the form

$$E^P \left[ \int_0^T U(t, Y_t, c_t) dt + \Phi(X_T) \right].$$

In Chapter 5 the factor process  $Y$  was assumed to be observable, but this is now completely changed by the following assumption.

**Assumption 11.0.1** *We cannot observe the process  $Y$  directly. The only information available us is the one generated by the technology process  $S$ . At time  $t$ , the information available to us is thus given by  $\mathcal{F}_t^S$  and, in particular, our controls must be adapted to the observable filtration  $\mathbf{F}^S$ .*

Given this assumption we note the following.

- Since we cannot observe  $Y$ , a local utility function of the form  $U(t, Y_t, c_t)$  no longer makes sense. Instead we have to replace it by a utility function of the form

$$E^P \left[ \int_0^T U(t, c_t) dt + \Phi(X_T) \right].$$

- Since we observe  $Y$  only indirectly, through observations of  $S$ , it seems intuitively clear that we must (somehow) estimate  $Y_t$  given the information contained in  $\mathcal{F}_t^S$ .
- We are thus led to the **filtering problem** of determining the conditional distribution  $\mathcal{L}(Y_t | \mathcal{F}_t^S)$  of  $Y_t$  conditional on  $\mathcal{F}_t^S$ .

In the next chapters the reader will therefore find a reasonably self contained introduction to non linear filtering theory.

## Chapter 12

# Non Linear Filtering Theory

In this chapter we will present some of the basic ideas and results of non linear filtering, which we need in order to study equilibrium models with partial observations. For the benefit of the interested reader we present some more advanced results in Appendix B.

### 12.1 The filtering model

We consider a filtered probability space  $(\Omega, \mathcal{F}, P, \mathbf{F})$  where as usual the filtration  $\mathbf{F} = \{\mathcal{F}_t; t \geq 0\}$  formalizes the idea of an increasing flow of information. The basic model of non linear filtering consists of a pair of processes  $(Y, Z)$  with dynamics as follows.

$$dY_t = a_t dt + dM_t, \quad (12.1)$$

$$dZ_t = b_t dt + dW_t. \quad (12.2)$$

In this model the processes  $a$  and  $b$  are allowed to be arbitrary  $\mathbf{F}$ -adapted processes,  $M$  is an  $\mathbf{F}$ -martingale, and  $W$  is an  $\mathbf{F}$ -Wiener process. At the moment we also assume that  $M$  and  $W$  are independent. We will below consider models where  $Y$  and  $Z$  are multidimensional, and where there is correlation between  $M$ , and  $W$  but for the moment we only consider this simple model.

**Remark 12.1.1** *Note that  $M$  is allowed to be an arbitrary martingale, so it does not have to be a Wiener process. The assumption that  $W$  is Wiener is, however, very important.*

The interpretation of this model is that we are interested in the **state process**  $Y$ , but that we cannot observe  $Y$  directly. What we can observe is instead the **observation process**  $Z$ , so our main problem is to draw conclusions about

$Y$ , given the observations of  $Z$ . We would for example like to compute the condition expectation

$$\hat{Y}_t = E [Y_t | \mathcal{F}_t^Z],$$

where  $\mathcal{F}_t^Z = \sigma \{Z_s, s \leq t\}$  is the information generated by  $Z$  on the time interval  $[0, t]$  or, more ambitiously, we would like to compute  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$ , i.e. the entire conditional distribution of  $Y_t$  given observations of  $Z$  on  $[0, t]$ .

A very common concrete example of the model above is given by a model of the form

$$\begin{aligned} dY_t &= \mu(t, Y_t)dt + \sigma(t, Y_t)dV_t, \\ dZ_t &= b(Y_t)dt + dW_t \end{aligned}$$

where  $V$  and  $W$  are independent Wiener processes. In this case we can thus observe  $Y$  indirectly through the term  $b(Y_t)$ , but the observations are corrupted by the noise generated by  $W$ .

## 12.2 The innovation process

Consider again the  $Z$ -dynamics

$$dZ_t = b_t dt + dW_t.$$

Our best guess of  $b_t$  given  $\mathcal{F}_t^Z$  is obviously given by  $\hat{b}_t = E [b_t | \mathcal{F}_t^Z]$ , and  $W$  is a process with zero mean so, at least intuitively, we expect that we would have

$$E [dZ_t | \mathcal{F}_t^Z] = \hat{b}_t dt.$$

This would imply that the “detrended” process  $\nu$ , defined by  $d\nu_t = dZ_t - \hat{b}_t dt$  should be an  $\mathbf{F}^Z$ -martingale. As we will see below, this conjecture is correct and we can even improve on it, but first the formal definition.

**Definition 12.2.1** *The innovation process  $\nu$  is defined by*

$$d\nu_t = dZ_t - \hat{b}_t dt.$$

We now have the following central result, which we will use repeatedly in connection with optimal investment models under partial information.

**Proposition 12.2.1** *The innovation process  $\nu$  is an  $\mathbf{F}^Z$ -Wiener process.*

**Proof.** We give a sketch of the proof. According to the Levy Theorem, it is enough to prove the following

- (i) The process  $\nu$  is an  $\mathbf{F}^Z$ -martingale.
- (ii) The process  $\nu_t^2 - t$  is an  $\mathbf{F}^Z$ -martingale.



To prove (i) we use the definition of  $\nu$  to obtain

$$E_s^Z [\nu_t - \nu_s] = E_s^Z [Z_t - Z_s] - E_s^Z \left[ \int_s^t \hat{b}_u du \right],$$

where we have used the shorthand notation

$$E_s^Z [\cdot] = E [\cdot | \mathcal{F}_s^Z].$$

From the  $Z$ -dynamics we have

$$Z_t - Z_s = \int_s^t b_u du + W_t - W_s,$$

so we can write

$$E_s^Z [\nu_t - \nu_s] = E_s^Z \left[ \int_s^t \{b_u - \hat{b}_u\} du \right] + E_s^Z [W_t - W_s].$$

Using iterated expectations and the  $\mathbf{F}$ -Wiener property of  $W$  we have

$$E_s^Z [W_t - W_s] = E_s^Z [E [W_t - W_s | \mathcal{F}_s]] = 0.$$

We also have

$$\begin{aligned} E_s^Z \left[ \int_s^t \{b_u - \hat{b}_u\} du \right] &= \int_s^t E_s^Z [b_u - \hat{b}_u] du = \int_s^t E_s^Z [E_u^Z [b_u - \hat{b}_u]] du \\ &= \int_s^t E_s^Z [\hat{b}_u - \hat{b}_u] du = 0, \end{aligned}$$

which proves (i).

To prove (ii) we use the Ito formula to obtain

$$d\nu_t^2 = 2\nu_t d\nu_t + (d\nu_t)^2.$$

Since  $d\nu_t = (b_t - \hat{b}_t) + dW_t$  we see that  $(d\nu_t)^2 = dt$ , so we have

$$d\nu_t^2 - dt = 2\nu_t d\nu_t.$$

From (i) we know that  $\nu$  is a martingale so the term  $2\nu_t d\nu_t$  should be a martingale increment, which proves (ii). ■

**Remark 12.2.1** *Note that we have, in a sense, cheated a little bit, since the proof of (ii) actually requires a stochastic calculus theory which covers stochastic integrals w.r.t. general martingales and not only Wiener processes. This is the case both when we use the Ito formula on  $\nu^2$  without knowing a priori that  $\nu$  is an Ito process, and the conclusion that  $\nu_t d\nu_t$  is a martingale increment without having an a priori guarantee that  $d\nu_t$  is a stochastic differential w.r.t. a Wiener process. Given this general stochastic calculus, the proof is completely correct.*

The innovation process  $\nu$  will play a very important role in the theory and in the economic applications, and to highlight this role we now reformulate the result above in a slightly different way.

**Proposition 12.2.2** *The  $Z$  dynamics can be written as*

$$dZ_t = \hat{b}_t dt + d\nu_t, \quad (12.3)$$

where  $\nu$  is an  $\mathbf{F}^Z$ -Wiener process.

**Remark 12.2.2** *Note that we now have two expressions for the  $Z$  dynamics. We have the original dynamics*

$$dZ_t = b_t dt + dW_t,$$

and we have

$$dZ_t = \hat{b}_t dt + d\nu_t,$$

it is now important to realize that the  $Z$  process in the left hand of these equations is, trajectory by trajectory, **exactly the same process**. The difference is that the first equation gives us the  $Z$ -dynamics relative to the filtration  $\mathbf{F}$ , whereas the second equation gives us the  $Z$ -dynamics w.r.t. the  $\mathbf{F}^Z$ -filtration.

### 12.3 Filter dynamics and the FKK equations

We now go on to derive an equation for the dynamics of the filter estimate  $\hat{Y}$ . From the  $Y$  dynamics, and from the previous argument concerning  $Z$ , the obvious guess is that the term  $d\hat{Y}_t - \hat{a}_t dt$  should be a martingale, and this is indeed the case.

**Lemma 12.3.1** *The process  $m$ , defined by*

$$dm_t = d\hat{Y}_t - \hat{a}_t dt,$$

is an  $\mathbf{F}^Z$ -martingale.

**Proof.** We have

$$\begin{aligned} E_s^Z [m_t - m_s] &= E_s^Z [Y_t - Y_s] - E_s^Z \left[ \int_s^t \hat{a}_u du \right] \\ &= E_s^Z \left[ \int_s^t \{a_u - \hat{a}_u\} du \right] + E_s^Z [M_t - M_s] \end{aligned}$$

We have

$$E_s^Z \left[ \int_s^t \{a_u - \hat{a}_u\} du \right] = E_s^Z \left[ \int_s^t E_s^Z [a_u - \hat{a}_u] du \right] = E_s^Z \left[ \int_s^t \{a_u - \hat{a}_u\} du \right] = 0,$$

and we also have

$$E_s^Z [M_t - M_s] = E_s^Z [E [M_t - M_s | \mathcal{F}_s]] = 0.$$

■

We thus have the filter dynamics

$$d\hat{Y}_t = \hat{a}_t dt + dm_t,$$

where  $m$  is an  $\mathbf{F}^Z$ -martingale, and it remains to see if we can say something more specific about  $m$ . From the definition of the innovation process  $\nu$  it seems reasonable to hope that we have the equality

$$\mathcal{F}_t^Z = \mathcal{F}_t^\nu, \quad (12.4)$$

and, if this conjecture is true, the Martingale Representation Theorem for Wiener processes would guarantee the existence of an adapted process  $h$  such that

$$dm_t = h_t d\nu_t.$$

The conjecture (12.4) is known as the “innovations hypothesis” and in its time it occupied a minor industry. In discrete time the corresponding innovations hypothesis is more or less trivially true, but in continuous time the situation is much more complicated. As a matter of fact, the (continuous time) innovations hypothesis is **not** generally true, but the good news is that Fujisaki, Kallianpur and Kunita proved the following result, which we quote from [16].

**Proposition 12.3.1** *There exists an adapted process  $h$  such that*

$$dm_t = h_t d\nu_t. \quad (12.5)$$

We thus have the filter dynamics

$$d\hat{Y}_t = \hat{a}_t dt + h_t d\nu_t, \quad (12.6)$$

and it remains to determine the precise structure of the **gain** process  $h$ . We have the following result.

**Proposition 12.3.2** *The gain process  $h$  is given by*

$$h_t = \widehat{Y}_t \widehat{b}_t - \widehat{Y}_t \widehat{b}_t \quad (12.7)$$

**Proof.** We give a slightly heuristic proof. The full formal proof uses the same idea as below, but it is more technical. We start by noticing that, for  $s < t$  we have (from iterated expectations)

$$E \left[ Y_t Z_t - \widehat{Y}_t Z_t \mid \mathcal{F}_s^Z \right] = 0$$

This leads to the heuristic identity

$$E \left[ d(YZ)_t - d(\widehat{Y}Z)_t \middle| \mathcal{F}_s^Z \right] = 0 \quad (12.8)$$

From the (general) Ito formula we have, using the independence between  $W$  and  $M$ ,

$$d(YZ)_t = Y_t b_t dt + Y_t dW_t + Z_t a_t dt + Z_t dM_t$$

The Ito formula applied to (12.6) and (12.3) gives us

$$d(\widehat{Y}Z)_t = \widehat{Y}_t \widehat{b}_t dt + \widehat{Y}_t d\nu_t + Z_t \widehat{a}_t dt + Z_t h d\nu_t + h dt,$$

where the term  $h dt$  comes from the equality  $(d\nu_t)^2 = dt$ , since  $\nu$  is a Wiener process. Plugging these expressions into the formula (12.8) gives us the expression

$$\left( \widehat{Y}_t \widehat{b}_t + Z_t \widehat{a}_t - \widehat{Y}_t \widehat{b}_t - Z_t \widehat{a}_t - h_t \right) dt = 0.$$

from which we conclude (12.7). ■ If we collect our findings we have the main result of nonlinear filtering, namely the Fujisaki-Kallianpur-Kunita filtering equations.

**Theorem 12.3.1 (The FKK Filtering Equations)** *The filtering equations are*

$$d\widehat{Y}_t = \widehat{a}_t dt + \left\{ \widehat{Y}_t \widehat{b}_t - \widehat{Y}_t \widehat{b}_t \right\} d\nu_t, \quad (12.9)$$

$$d\nu_t = dZ_t - \widehat{b}_t dt. \quad (12.10)$$

A simple calculation shows that we can write the gain process  $h$  as

$$h_t = E \left[ \left( Y_t - \widehat{Y}_t \right) \left( b_t - \widehat{b}_t \right) \middle| \mathcal{F}_t^Z \right], \quad (12.11)$$

so we see that the innovations are amplified by the conditional error covariance between  $Y$  and  $b$ .

## 12.4 The general FKK equations

We now extend our filtering theory to a more general model.

**Assumption 12.4.1** *We consider a filtered probability space  $\{\Omega, \mathcal{F}, P, \mathbf{F}\}$ , carrying a martingale  $M$  and a Wiener process  $W$ , where  $M$  and  $W$  are not assumed to be independent. On this space we have the model*

$$dY_t = a_t dt + dM_t, \quad (12.12)$$

$$dZ_t = b_t dt + \sigma_t dW_t, \quad (12.13)$$

where  $a$  and  $b$  are  $\mathbf{F}$  adapted scalar processes. The process  $\sigma$  is assumed to be strictly positive and  $\mathbf{F}^Z$  adapted.

We note that the assumption about  $\sigma$  being  $\mathbf{F}^Z$  adapted is not so much an assumption as a result, since the quadratic variation property of  $W$  implies that we can in fact estimate  $\sigma_t^2$  without error on an arbitrary short interval. The moral of this is that although the drift  $b$  will typically depend in some way on the state process  $Y$ , we can **not** let  $\sigma$  be of the form  $\sigma_t = \sigma(Y_t)$ , since then the filter would trivialize. We also note that we cannot allow  $\sigma$  to be zero at any point, since then the filter will degenerate.

In a setting like this it is more or less obvious that the natural definition of the innovation process  $\nu$  is by

$$d\nu_t = \frac{1}{\sigma_t} \{dZ_t - \pi_t(b) dt\},$$

and it is not hard to prove that  $\nu$  is a Wiener process. We can now more or less copy the arguments in Section 12.3, and after some work we end up with the following general FKK equations.

**Theorem 12.4.1 (Fujisaki-Kallianpur-Kunita)** *With assumptions as above we have the following filter equations.*

$$\begin{aligned} d\hat{Y}_t &= \hat{a}_t dt + \left[ \hat{D}_t + \frac{1}{\sigma_t} \{ \widehat{Y_t b_t} - \hat{Y}_t \hat{b}_t \} \right] d\nu_t, \\ d\nu_t &= \frac{1}{\sigma_t} \{ dZ_t - \hat{b}_t dt \}. \end{aligned}$$

where

$$D_t = \frac{d\langle M, W \rangle_t}{dt}$$

Furthermore, the innovation process  $\nu$  is an  $\mathbf{F}^Z$  Wiener process.

A proper definition of the process  $D$  above requires a more general theory for semimartingales, but for most applications the following results are sufficient.

- If  $M$  has continuous trajectories, then

$$dD_t = dM \cdot dW_t,$$

with the usual Ito multiplication rules.

- If  $M$  is a pure jump process without a Wiener component, then

$$dD_t = 0.$$

## 12.5 Filtering a Markov process

A natural class of filtering problems to study is obtained if we consider a time homogeneous Markov process  $Y$ , living on some state space  $\mathcal{M}$ , with generator  $\mathcal{G}$ , and we are interested in estimating  $f(Y_t)$  for some real valued function  $f : \mathcal{M} \rightarrow R$ .

### 12.5.1 The Markov filter equations

If  $f$  is in the domain of  $\mathcal{G}$  we can then apply the Dynkin Theorem and obtain the dynamics

$$df(Y_t) = (\mathcal{G}f)(Y_t)dt + dM_t, \quad (12.14)$$

where  $M$  is a martingale.

**Remark 12.5.1** *For the reader who is unfamiliar with Dynkin and general Markov processes we note that a typical example would be that  $Y$  is governed by an SDE of the form*

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t^0$$

where  $W^0$  is a Wiener process. In this case the Ito formula will give us

$$df(Y_t) = \left\{ \mu(Y_t)f'(Y_t) + \frac{1}{2}\sigma^2(Y_t)f''(Y_t) \right\} dt + \sigma(Y_t)f'(Y_t)dW_t^0$$

so in this case the generator  $\mathcal{G}$  is given by

$$\mathcal{G} = \mu(y)\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2}{\partial y^2}.$$

and

$$dM_t = \sigma(Y_t)f'(Y_t)dW_t^0.$$

Let us now assume that the observations are of the form

$$dZ_t = b(Y_t)dt + dW_t,$$

where  $b$  is a function  $b : \mathcal{M} \rightarrow R$ , and  $W$  is a Wiener process which, for simplicity, is independent of  $Y$ . The filtration  $\mathbf{F}$  is defined as

$$\mathcal{F}_t = \mathcal{F}_t^Y \vee \mathcal{F}_t^W.$$

We thus have the filtering model

$$\begin{aligned} df(Y_t) &= (\mathcal{G}f)(Y_t)dt + dM_t, \\ dZ_t &= b(Y_t)dt + dW_t, \end{aligned}$$

and if we introduce the notation

$$\pi_t(g) = E[g(Y_t) | \mathcal{F}_t^Z]. \quad (12.15)$$

for any real valued function  $g : \mathcal{M} \rightarrow R$ , we can apply the FKK equations to obtain

$$d\pi_t(f) = \pi_t(\mathcal{G}f)dt + \{\pi_t(fb) - \pi_t(f) \cdot \pi_t(b)\}d\nu_t, \quad (12.16)$$

$$d\nu_t = dZ_t - \pi_t(b)dt. \quad (12.17)$$

### 12.5.2 On the filter dimension

We would now like to consider the equation (12.16) as an SDE driven by the innovation process  $\nu$ , but the problem is that the equation is not closed, since in the SDE for  $\pi_t(f)$  we have the expressions  $\pi_t(\mathcal{G}f)$ ,  $\pi_t(fb)$ , and  $\pi_t(b)$  which all have to be determined in some way.

The obvious way to handle, for example, the term  $\pi_t(b)$ , is of course the following

- Use Dynkin on  $b$  to obtain

$$db(Y_t) = (\mathcal{G}b)(Y_t)dt + dM_t^b,$$

where  $M^b$  is a martingale.

- Apply the FKK equations to the system

$$\begin{aligned} db(Y_t) &= (\mathcal{G}b)(Y_t)dt + dM_t^b, \\ dZ_t &= b(Y_t)dt + dW_t, \end{aligned}$$

to obtain the filter equation

$$d\pi_t(b) = \pi_t(\mathcal{G}b) dt + \{\pi_t(b^2) - (\pi_t(b))^2\} d\nu_t$$

We now have an equation for  $d\pi_t(b)$ , but this equation contains (among other things) the term  $\pi_t(b^2)$ , so we now need an equation for this term. In order to derive that equation, we can of course apply Dynkin to the process  $b^2(Y_t)$  and again use the FKK equations, but this leads to the equation

$$d\pi_t(b^2) = \pi_t(\mathcal{G}b^2) dt + \{\pi_t(b^3) - \pi_t(b^2) \cdot \pi_t(b)\} d\nu_t$$

and we now have to deal with the term  $\pi_t(b^3)$  etc.

As the reader realizes, this procedure will in fact lead to an infinite number of filtering equations. This is in fact the generic situation for a filtering problem, and the argument is roughly as follows.

- In general there will not exist a finite dimensional sufficient statistic for the  $Y$  process.
- In particular, an old estimate  $\pi_t(Y)$ , plus the new information  $d\nu_t$  is not sufficient to allow us to determine an updated estimate  $\pi_{t+dt}(Y)$ .
- In order to be able to update, even such a simple object as the conditional expectation  $\pi_t(Y)$ , we will, in the generic case, need the **entire conditional distribution**  $\mathcal{L}(Y_t|\mathcal{F}_t^Z)$ .
- The conditional distribution is typically an infinite dimensional object.
- In the generic case we can therefore expect to have an **infinite dimensional filter**.

An alternative way of viewing (12.16) is now to view it, not as a scalar equation for a fixed choice of  $f$ , but rather as an infinite number of equations, with one equation for each (say, bounded continuous)  $f$ . Viewed in this way, the filtering equation (12.16) represents an infinite dimensional system for the determination of the entire condition distribution  $\mathcal{L}(Y_t|\mathcal{F}_t^Z)$ . We will capitalize on this idea later when we derive the dynamics for the conditional density.

### 12.5.3 Finite dimensional filters

From the discussion above it is now clear that the generic filter is infinite dimensional, the reason being that we need the entire conditional distribution of  $Y$  in order to update our filter estimates, and this distribution is typically an infinite dimensional object. This is, in some sense, bad news, but there is really nothing we can do about the situation - it is simply a fact of life.

We can, however, also draw some more positive conclusions from the dimension argument, and after a moments reflection we have the following important idea.

**Idea 12.5.1** *If we know on a priori grounds that, for all  $t$ , the conditional distribution  $\mathcal{L}(Y_t|\mathcal{F}_t^Z)$  belongs to a class of probability distributions which is parameterized by a **finite number of parameters**, then we can expect the have a finite dimensional filter. The filter equations should then provide us with the dynamics of the parameters for the conditional distribution.*

There are in fact two well known models when we have a priori information of the type above. They are known as the **Kalman** model and the **Wonham** model respectively. We will discuss them in detail later, but we introduce them already at this point.

#### The Kalman model:

The simplest case of a Kalman model is given by the linear system

$$\begin{aligned} dY_t &= aY_t dt + cdW_t^0, \\ dZ_t &= bY_t dt + dW_t, \end{aligned}$$

where  $W^0$  and  $W$  are, possibly correlated, Wiener processes. For this model it is easy to see that the pair  $(Y, Z)$  will be jointly Gaussian. We then recall the standard fact that if  $(\xi, \eta)$  is a pair of Gaussian vectors, then the conditional distribution  $\mathcal{L}\{\xi | \eta\}$  will also be Gaussian. This property can be shown to extend also to the process case, so we conclude that the conditional distribution  $\mathcal{L}(Y_t|\mathcal{F}_t^Z)$  is Gaussian. The Gaussian distribution is, however, determined by only two parameters - the mean and the variance, so for this model we expect to have a two dimensional filter with one equation for the conditional mean and another for the conditional variance.



**The Wonham model:**

In the Wonham model, the process  $Y$  is a continuous time Markov chain which takes values in the finite state space  $\{1, 2, \dots, n\}$ , and where the observation process is of the form

$$dZ_t = b(Y_t)dt + dW_t.$$

For this model it is immediately obvious that the conditional distribution  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$  is determined by a finite number of parameters, since it is in fact determined by the conditional probabilities  $p_t^1, \dots, p_t^n$  where  $p_t^i = P(Y_t = i | \mathcal{F}_t^Z)$ . We thus expect to have an  $n$ -dimensional filter.

## 12.6 The Kalman filter

In this section we will discuss the Kalman filter in some detail. We will present the full multidimensional Kalman model and provide the

### 12.6.1 The Kalman model

The basic Kalman model is as follows.

$$\begin{aligned} dY_t &= AY_t dt + CdW_t^0, \\ dZ_t &= BY_t dt + DdW_t, \end{aligned}$$

All processes are allowed to be vector valued with  $Y_t \in R^k$ ,  $Z_t \in R^k$ ,  $W_t \in R^n$ , and  $W^0 \in R^d$ . The matrices  $A$ ,  $C$ ,  $B$  and  $D$  have the obvious dimensions, and we need two basic assumptions.

**Assumption 12.6.1** *We assume the following.*

1. *The  $n \times n$  matrix  $D$  is invertible.*
2. *The Wiener processes  $W$  and  $W^0$  are independent.*
3. *The distribution of  $Y_0$  is Gaussian with mean vector  $y_0$  and covariance matrix  $R_0$ .*

The independence assumption, and also the Gaussian assumption concerning  $Y_0$ , can be relaxed, but the invertibility of  $D$  is important and cannot be omitted.

This model can of course be treated using the FKK theory, but since the vector valued case is a bit messy we only carry out the derivation in detail for the simpler scalar case in Section 12.6.2. In Section 12.6.3 we then state the general result without proof.

### 12.6.2 Deriving the filter equations in the scalar case

In this section we will derive the filter equations, but for the general vector valued Kalman model this turns out to be a bit messy. We therefore confine ourselves to deriving the filter in the special case of a scalar model of the form

$$\begin{aligned} dY_t &= aY_t dt + cdW_t^0, \\ dZ_t &= Y_t dt + dW_t, \end{aligned}$$

where all processes and constants are scalar. From the discussion in Section 12.5.3 we know that the conditional distribution  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$  is Gaussian, so it should be enough to derive filter equations for the conditional mean and variance.

The FKK equation for the conditional mean is given by

$$d\pi_t(Y) = a\pi_t(Y) dt + \left\{ \pi_t(Y^2) - (\pi_t(Y))^2 \right\} d\nu_t, \quad (12.18)$$

$$d\nu_t = dZ_t - b\pi_t(Y) dt. \quad (12.19)$$

This would be a closed system if we did not have the term  $\pi_t(Y^2)$ . Along the lines of the discussion in Section 12.5.3 we therefore use Ito on the process  $Y^2$  to obtain

$$dY_t^2 = \{2aY_t^2 + c^2\} dt + 2cY_t dW_t^0.$$

This will give us the filter equation

$$d\pi_t(Y^2) = \{2a\pi_t(Y^2) + c^2\} dt + \{\pi_t(Y^3) - \pi_t(Y^2) \cdot \pi_t(Y)\} d\nu_t. \quad (12.20)$$

We now have the term  $\pi_t(Y^3)$  to deal with, and a naive continuation of the procedure above will produce an infinite number of filtering equations for all conditional moments  $\pi_t(Y^k)$ , where  $k = 1, 2, \dots$ .

In this case, however, because of the particular dynamical structure of the model, we know that the conditional distribution  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$  is Gaussian. We therefore define the conditional variance process  $H$  by

$$H_t = E \left[ \left( Y_t - \widehat{Y}_t \right)^2 \middle| \mathcal{F}_t^Z \right] = \pi_t(Y^2) - (\pi_t(Y))^2.$$

In order to obtain a dynamical equation for  $H$  we apply Ito to (12.18) to obtain

$$d(\pi_t(Y))^2 = \{2a(\pi_t(Y))^2 + H_t^2\} dt + 2\pi_t(Y) H_t d\nu_t.$$

Using this and equation (12.20) we obtain the  $H$  dynamics as

$$dH_t = \{2aH_t + c^2 - H_t^2\} dt + \{\pi_t(Y^3) - 3\pi_t(Y^2) \pi_t(Y) + 2(\pi_t(Y))^3\} d\nu_t$$

We can now use the Gaussian structure of the problem and recall that for any Gaussian variable  $\xi$  we have

$$E[\xi^3] = 3E[\xi^2]E[\xi] - 2(E[\xi])^3.$$

Since  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$  is Gaussian we thus conclude that

$$\pi_t(Y^3) - 3\pi_t(Y^2)\pi_t(Y) + 2(\pi_t(Y))^3 = 0,$$

so, as expected,  $H$  is in fact deterministic and we have the Kalman filter

$$\begin{aligned} d\hat{Y}_t &= a\hat{Y}_t dt + H_t d\nu_t, \\ \dot{H}_t &= 2aH_t + c^2 - H_t^2, \\ d\nu_t &= dZ_t - \hat{Y}_t dt. \end{aligned}$$

The first equation gives us the evolution of the conditional mean. The second equation, which is a so called Riccati equation, gives us the evolution of the conditional variance. We also note that since the conditional distribution is Gaussian, the Kalman filter does in fact provide us with the entire conditional distribution, and not just the conditional mean and variance.

### 12.6.3 The full Kalman model

We now return to the vector model

$$\begin{aligned} dY_t &= AY_t dt + C dW_t^0, \\ dZ_t &= BY_t dt + D dW_t. \end{aligned}$$

This model can be treated very much along the lines of the scalar case in the previous section, but the calculations are a bit more complicated. We thus confine ourselves to stating the final result.

**Proposition 12.6.1 (The Kalman Filter)** *With notation and assumptions as above we have the following filter equations where ' denote transpose.*

$$d\hat{Y}_t = a\hat{Y}_t dt + R_t B'(DD')^{-1} d\nu_t, \quad (12.21)$$

$$\dot{R}_t = AR_t + R_t A' - R_t B'(DD')^{-1} B R_t + CC', \quad (12.22)$$

$$d\nu_t = dZ_t - B\hat{Y}_t dt. \quad (12.23)$$

Furthermore, the conditional error covariance matrix is given by  $R$  above.

## 12.7 The Wonham filter

We consider again the Wonham model. In this model, the  $Y$  process is a time homogeneous Markov chain on a finite state space  $D$ , and without loss of generality we may assume that  $D = \{1, 2, \dots, n\}$ . We denote the intensity matrix of  $Y$  by  $H$ , and the probabilistic interpretation is that

$$\begin{aligned} P(Y_{t+h} = j | Y_t = i) &= H_{ij}h + o(h), \quad i \neq j, \\ H_{ii} &= -\sum_{j \neq i} H_{ij}. \end{aligned}$$

We cannot observe  $Y$  directly, but instead we can observe the process  $Z$ , defined by

$$dZ_t = b(Y_t) + dW_t, \quad (12.24)$$

where  $W$  is a Wiener process. In this model it is obvious that the conditional distribution of  $Y$  will be determined by the conditional probabilities, so we define the indicator processes  $\delta_t^1, \dots, \delta_t^n$  by

$$\delta_i(t) = I\{Y_t = i\}, \quad i = 1, \dots, n,$$

where  $I\{A\}$  denotes the indicator for an event  $A$ , so  $I\{A\} = 1$  if  $A$  occurs, and zero otherwise. The Dynkin Theorem, and a simple calculation gives us the equation

$$d\delta_i(t) = \sum_{j=1}^n H_{ji} \delta_j(t) dt + dM_t^i, \quad i = 1, \dots, n, \quad (12.25)$$

where  $M^1, \dots, M^n$  are martingales. On vector form this we can thus write

$$d\delta(t) = H'\delta(t)dt + dM_t.$$

Applying the FKK Theorem to the dynamics above gives us the filter equations

$$\begin{aligned} d\pi_t(\delta_i) &= \sum_{j=1}^n H_{ji} \pi_t(\delta_j) dt + \{\pi_t(\delta_i b) - \pi_t(\delta_i) \pi_t(b)\} d\nu_t, \quad i = 1, \dots, n, \\ d\nu_t &= dZ_t - \pi_t(b) dt \end{aligned}$$

We now observe that, using the notation  $b_i = b(i)$ , we have the obvious relations

$$b(Y_t) = \sum_{j=1}^n b_j \delta_j(t), \quad \delta_i(t) b(Y_t) = \delta_i(t) b_i.$$

which gives us

$$\pi_t(b) = \sum_{j=1}^n b_j \hat{\delta}_j(t), \quad \pi_t(\delta^i b) = \hat{\delta}_i(t) b_i.$$

Plugging this into the FKK equations gives us the Wonham filter.

**Proposition 12.7.1 (The Wonham Filter)** *With assumptions as above, the Wonham filter is given by*

$$d\hat{\delta}_i(t) = \sum_{j=1}^n H_{ji} \hat{\delta}_j(t) dt + \left\{ b_i \hat{\delta}_i(t) - \hat{\delta}_i(t) \cdot \sum_{j=1}^n b_j \hat{\delta}_j(t) \right\} d\nu_t, \quad (12.26)$$

$$d\nu_t = dZ_t - \sum_{j=1}^n b_j \hat{\delta}_j(t) dt, \quad (12.27)$$

where

$$\hat{\delta}_i(t) = P(Y_t = i | \mathcal{F}_t^Z), \quad i = 1, \dots, n.$$

## 12.8 Exercises

**Exercise 12.1** Consider the filtering model

$$\begin{aligned}dX_t &= a_t dt + dV_t \\dZ_t &= b_t dt + \sigma_t dW_t\end{aligned}$$

where

- The process  $\sigma$  is  $\mathcal{F}_t^Z$  adapted and positive.
- $W$  and  $V$  are, possibly correlated, Wiener processes.

Prove, along the lines in the lecture notes, that the filtering equations are given by

$$\begin{aligned}d\hat{X}_t &= \hat{a}_t dt + \left[ \hat{D}_t + \frac{1}{\sigma_t} \left\{ \widehat{X_t b_t} - \hat{X}_t \hat{b}_t \right\} \right] d\nu_t \\d\nu_t &= \frac{1}{\sigma_t} \left\{ dZ_t - \hat{b}_t dt \right\} \\D_t &= \frac{d\langle V, W \rangle_t}{dt}\end{aligned}$$

**Exercise 12.2** Consider the filtering model

$$dZ_t = X dt + dW_t$$

where  $X$  is a random variable with distribution function  $F$ , and  $W$  is a Wiener process which is independent of  $X$ . As usual we observe  $Z$ . Write down the infinite system of filtering equations for the determination of  $\Pi_t[X] = E[X | \mathcal{F}_t^Z]$ .

## 12.9 Notes

The original paper [16] provides a very readable account of the FKK theory. The two volume set [28] is a standard reference on filtering. It includes, apart from the Wiener driven FKK framework, also a deep theory of point processes and the related filtering theory. It is, however, not an easy read. A far reaching account of Wiener driven filtering theory is given in [1].



## Chapter 13

# Production Equilibrium under Partial Information

In this chapter we will study a simple version of the CIR model, with the difference that the factor process cannot be observed directly.

### 13.1 The model

We assume the existence of a scalar production technology (with the usual interpretation) with dynamics given by

$$dS_t = Y_t S_t dt + S_t \sigma dW_t^s, \quad (13.1)$$

where  $W^s$  is Wiener. The scalar factor process  $Y$ , determining the rate of return on physical investment, is assumed to have dynamics given by

$$dY_t = (AY_t + B)dt + CdW_t^y, \quad (13.2)$$

where  $W^y$  is a Wiener process. For notational simplicity we assume that  $W^s$  and  $W^y$  are independent. The filtration generated by  $W^s$  and  $W^y$  is denoted by  $\mathbf{F}$ , so  $\mathcal{F}_t = \sigma\{W_u^s, W_u^y; 0 \leq u \leq t\}$ .

We consider a representative agent who only has access to the information generated by observations of the  $S$  process, so all his actions must be adapted to the filtration  $\mathbf{F}^S$ , where  $\mathcal{F}_t^S = \sigma\{S_u; 0 \leq u \leq t\}$ . The agent can invest in the following assets.

- The physical production process  $S$ .
- A risk free asset  $B$  in zero net supply with dynamics

$$dB_t = r_t B_t dt,$$

where the  $\mathbf{F}^S$ -adapted risk free rate of return  $r$  will be determined in equilibrium.

The controls available to the agent are the consumption rate  $c$  and the portfolio weight  $u$  on physical investment and both of these are required to be  $\mathbf{F}^S$ -adapted. The object of the agent is to maximize expected utility of the form

$$E^P \left[ \int_0^T U(t, c_t) dt \right].$$

### 13.2 Projecting the $S$ dynamics.

We define the process  $Z$  by

$$dZ_t = \frac{dS_t}{\sigma S_t} \quad (13.3)$$

and we note that  $\mathbf{F}^Z = \mathbf{F}^S$ . We can write the observation dynamics as

$$dZ_t = \frac{Y_t}{\sigma} dt + dW_t^s.$$

and the innovations process  $\nu$  is now defined as usual by

$$d\nu_t = dZ_t - \frac{\hat{y}_t}{\sigma} dt, \quad (13.4)$$

From proposition 12.2.1 we know that  $\nu$  is an  $\mathbf{F}^S$  Wiener process. Plugging (13.3) into (13.4) gives us the  $S$ -dynamics projected onto the observable filtration  $\mathbf{F}^S$  as

$$dS_t = \hat{y}_t S_t dt + \sigma S_t d\nu_t. \quad (13.5)$$

### 13.3 The filtering equations

Recalling the dynamics of the pair  $(Y, Z)$  we have

$$\begin{aligned} dY_t &= (AY_t + B)dt + C dW_t^y, \\ dZ_t &= \frac{Y_t}{\sigma} dt + dW_t^s, \end{aligned}$$

and we recognize this as a standard Kalman model. We thus have the Kalman filter equations

$$d\hat{y}_t = (A\hat{y}_t + B) + H_t d\nu_t,$$

with the innovations process  $\nu$  defined by (13.4), and the gain  $H$  satisfying a Riccati equation.

### 13.4 The control problem

In order to formulate the partially observed control problem, we start by deriving the relevant portfolio dynamics, but first we introduce a Markovian assumption concerning the risk free rate.



**Assumption 13.4.1** We assume that the risk free rate process  $r$  is of the form

$$r_t = r(t, X_t, \hat{y}_t)$$

where  $X$  denotes portfolio value and  $r(t, x, y)$  is a deterministic function.

From (13.5) and from standard theory we see that the portfolio value dynamics are given by

$$dX_t = u_t X_t (\hat{y}_t - r_t) dt + (r_t X_t - c_t) dt + u_t X_t \sigma d\nu_t.$$

where  $u$  is the weight on the risky asset.

We are thus ready to state the control problem. The object is to maximize the expected utility

$$E^P \left[ \int_0^T U(t, c_t) dt \right].$$

over  $\mathbf{F}^S$ -adapted controls  $(c, u)$ , given the system

$$dX_t = u_t X_t (\hat{y}_t - r_t) dt + (r_t X_t - c_t) dt + u_t X_t \sigma d\nu_t, \quad (13.6)$$

$$d\hat{y}_t = (A\hat{y}_t + B) + H_t d\nu_t. \quad (13.7)$$

and the constraint  $c_t \geq 0$ . This, however, is a **standard problem with full information** so we can apply DynP in a standard manner. Denoting the optimal value function by  $V(t, x, y)$  we have the following HJB equation.

$$\begin{cases} V_t(t, x, y) + \sup_{c, u} \{U(t, c) + \mathbf{A}^{c, u} V(t, x, y)\} = 0, \\ V(T, x) = 0, \end{cases} \quad (13.8)$$

where the operator  $\mathbf{A}^{c, u}$  is defined as

$$\begin{aligned} \mathbf{A}^{c, u} V &= u(y - r)xV_x + (rx - c)V_x + \frac{1}{2}u^2 x^2 \sigma^2 V_{xx} \\ &+ (Ay + B)V_y + \frac{1}{2}H^2 V_{yy} + ux\sigma H V_{xy}. \end{aligned}$$

Assuming an interior optimum, we have the first order conditions

$$\begin{aligned} U'_c &= V_x, \\ \hat{u} &= \frac{r - y}{\sigma^2} \left( \frac{V_x}{xV_{xx}} \right) - \frac{H}{\sigma} \left( \frac{V_{xy}}{xV_{xx}} \right). \end{aligned}$$

## 13.5 Equilibrium

Since the risk free asset is in zero net supply, the equilibrium condition is  $\hat{u} = 1$ . Inserting this into the first order condition above we obtain the main result.

**Proposition 13.5.1** *The risk free rate and the Girsanov kernel  $\varphi$  are given by*

$$r(t, x, y) = y + \frac{xV_{xx}}{V_x}\sigma^2 + \frac{V_{xy}}{V_x}H\sigma, \quad (13.9)$$

$$\varphi(t, x, y) = \frac{xV_{xx}}{V_x}\sigma + \frac{V_{xy}}{V_x}H. \quad (13.10)$$

It is instructive to compare this result to the result we would have obtained if the factor process  $Y$  had been observable. There are similarities as well as differences. At first sight it may seem that the only difference is that  $Y$  is replaced by  $\hat{y}$ , but the situation is in fact a little bit more complicated than that.

- For the fully observable model the  $(S, Y)$  dynamics are of the form

$$\begin{aligned} dS_t &= Y_t S_t dt + S_t \sigma dW_t^s, \\ dY_t &= (AY_t + B)dt + C dW_t^y, \end{aligned}$$

where  $W^s$  and  $W^y$  are independent.

- For the partially observable model, the process  $Y$  is replaced by the filter estimate  $\hat{y}$ , and the  $(S, \hat{y})$  dynamics are of the form

$$\begin{aligned} dS_t &= \hat{y}_t S_t dt + \sigma S_t d\nu_t, \\ d\hat{y}_t &= (A\hat{y}_t + B) + H_t d\nu_t. \end{aligned}$$

Firstly we note that whereas  $S$  and  $Y$  are driven by independent Wiener processes,  $S$  and  $\hat{y}$  are driven by **the same** Wiener process, namely the innovation  $\nu$ . Secondly we note that the diffusion term  $C$  in the  $Y$  dynamics is replaced by  $H$  in the  $\hat{y}$  dynamics.

- Using Proposition 13.5.1 above, as well as Proposition 5.4.1 we see that the formulas for the short rate in the observable and the partially observable case are given as follows.

$$\begin{aligned} r(t, x, y) &= y + \frac{xV_{xx}}{V_x}\sigma^2, \\ r(t, x, \hat{y}) &= \hat{y} + \frac{xV_{xx}}{V_x}\sigma^2 + \frac{V_{xy}}{V_x}H\sigma. \end{aligned}$$

Apart from the fact that  $y$  refers to  $Y$  in the first formula and to  $\hat{y}$  in the second one, there are two differences between these formulas. Firstly, there is no mixed term in the completely observable model. We would perhaps have expected a term of the form

$$\frac{V_{xy}}{V_x}C\sigma$$

but this term vanishes because of the assumed independence between  $W^s$  and  $W^y$ . Secondly, the function  $V$  is not the same in the two formulas. We recall that  $V$  is the solution to the HJB equation, and this equation differs slightly between the two models.

## 13.6 Notes

The model studied above is a slightly simplified version of the model in [17]. It is also a special case of the model in [10], where a Kalman model with non-Gaussian initial data is analyzed.



# Chapter 14

## Endowment Equilibrium under Partial Information

In this chapter we study a partially observable version of the endowment model of Chapter 7.

### 14.1 The model

The main assumptions are as follows.

**Assumption 14.1.1** *We assume the existence of an endowment process  $e$  of the form*

$$de_t = a_t dt + b_t dW_t. \quad (14.1)$$

*Furthermore, we assume the following.*

- *The observable filtration is given by  $\mathbf{F}^e$ , i.e. all observations are generated by the endowment process  $e$ .*
- *The process  $a$  is **not** assumed to be observable, so it is not adapted to  $\mathbf{F}^e$ .*
- *The process  $b$  is adapted to  $\mathbf{F}^e$ .*
- *The process  $b$  is assumed to satisfy the non-degeneracy condition*

$$b_t > 0, \quad P\text{-a.s. for all } t. \quad (14.2)$$

Apart from these assumptions, the setup is that of Chapter 7, so we assume that there exists a risky asset  $S$  in unit net supply, giving the holder the right to the endowment  $e$ . We also assume the existence of a risk free asset in zero net supply. The initial wealth of the representative agent is assumed to equal  $S_0$  so

the agent can afford to buy the right to the endowment  $e$ . The representative agent is as usual assumed to maximize utility of the form

$$E \left[ \int_0^T U(t, c_t) dt \right].$$

## 14.2 Projecting the $e$ -dynamics

As usual for partially observable models, we start by projecting the relevant process dynamics onto the observable filtration. We thus define the process  $Z$  by

$$dZ_t = \frac{de_t}{b_t} \quad (14.3)$$

so that

$$dZ_t = \frac{a_t}{b_t} dt + dW_t,$$

and define the innovation process  $\nu$  as usual by

$$d\nu_t = dZ_t - \frac{\hat{a}_t}{b_t} dt,$$

where

$$\hat{a}_t = E[a_t | \mathcal{F}_t^e]$$

This gives us the  $Z$  dynamics on the  $\mathbf{F}^e$  filtration as

$$dZ_t = \frac{\hat{a}_t}{b_t} dt + d\nu_t,$$

and plugging this into (14.3) gives us the  $e$  dynamics projected onto the  $\mathbf{F}^e$  filtration as

$$de_t = \hat{a}_t dt + b_t d\nu_t. \quad (14.4)$$

## 14.3 Equilibrium

Given the formula (14.4) we are now back in a completely observable model, so we can quote Proposition 7.4.1 to obtain the main result.

**Proposition 14.3.1** *For the partially observed model above, the following hold.*

- The equilibrium short rate process is given by

$$r_t = - \frac{U_{ct}(t, e) + \hat{a}_t U_{cc}(t, e_t) + \frac{1}{2} \|b_t\|^2 U_{ccc}(t, e_t)}{U_c(t, e_t)}. \quad (14.5)$$

- The Girsanov kernel is given by

$$\varphi_t = \frac{U_{cc}(t, e_t)}{U_c(t, e_t)} \cdot b_t. \quad (14.6)$$

In an abstract sense we have thus completely solved the problem of equilibrium within a partially observable endowment model. In order to obtain more concrete results, we need to impose some extra structure, and this will be done in the next section.

## 14.4 A factor model

In this section we specialize the model above to a factor model of the form

$$\begin{aligned} de_t &= a(e_t, Y_t)dt + b(e_t)dW_t^e, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t^y, \end{aligned}$$

where, for simplicity, we assume that  $W^e$  and  $W^y$  are independent. Note that we cannot allow  $b$  to depend on the factor  $Y$ . We also assume log utility, so that

$$U(t, c) = e^{-\delta t} \ln(c).$$

As in Section 7.4.3 we easily obtain

$$\begin{aligned} r_t &= \delta + \frac{\hat{a}_t}{e_t} - \frac{b^2(e_t)}{e_t^2}, \\ \varphi_t &= -\frac{b(e_t)}{e_t}. \end{aligned}$$

where

$$\hat{a}_t = E[a(e_t, Y_t) | \mathcal{F}_t^e].$$

Given these expressions it is natural to specialize to the case when

$$\begin{aligned} a(e, y) &= e \cdot a(y), \\ b(e) &= b \cdot e, \end{aligned}$$

where  $b$  is a constant. This gives us

$$\begin{aligned} r_t &= \delta + \hat{a}_t - b^2, \\ \varphi_t &= -b. \end{aligned}$$

In order to obtain a finite filter for  $\hat{a} = E[a(Y_t) | \mathcal{F}_t^e]$  it is now natural to look for a Kalman model and our main result is as follows.

**Proposition 14.4.1** *Assume a model of the form*

$$\begin{aligned} de_t &= ae_t Y_t dt + be_t dW_t^e, \\ dY_t &= BY_t dt + C dW_t^y, \end{aligned}$$

*The risk free rate and the Girsanov kernel are then given by*

$$r_t = \delta - b^2 + a\hat{y}_t, \tag{14.7}$$

$$\varphi_t = -b. \tag{14.8}$$

*where  $\hat{y}$  is given by the Kalman filter*

$$d\hat{y} = B\hat{y}_t + H_t d\nu_t.$$





# Appendix A

## Basic Arbitrage Theory

### A.1 Portfolios

In this appendix we recall some central concepts and result from general arbitrage theory. For details the reader is referred to [2] or any other standard textbook on the subject.

We consider a market model consisting of  $N + 1$  financial assets (without dividends). We assume that the market is perfectly liquid, that there is no credit risk, no bid-ask spread, and that prices are not affected by our portfolios.

We are given a filtered probability space  $(\Omega, \mathcal{F}, P, \mathbf{F})$ , and by  $S_t^i$  we denote the price at time  $t$  of one unit of asset No.  $i$ , for  $i = 0, \dots, N$ . We let  $S$  denote the corresponding  $N$  dimensional column vector process, and all asset price processes are assumed to be adapted. The asset  $S^0$  will play a special role below as the **numeraire asset** and we assume that  $S^0$  is strictly positive with probability one. In general the price processes are allowed to be semimartingales but in this chapter we only consider the simpler case when the prices are driven by finite number of Wiener processes.

We now go on to define the concept of a “self financing portfolio”. Intuitively this is a portfolio strategy whee there is no external withdrawal from, or infusion of money to, the portfolio. It is far from trivial how this should be formalized in continuous time, but a careful discretization argument leads to the following formal definition, where we let  $h_t^i$  denote the number of units of asset No.  $i$  which are held in the portfolio.

**Definition A.1.1** *A portfolio strategy is an  $N+1$  dimensional adapted (row vector) process  $h = (h^1, \dots, h^N)$ . For a given strategy  $h$ , the corresponding value process  $X^h$  is defined by*

$$X_t^h = \sum_{i=0}^N h_t^i S_t^i = h_t S_t. \quad (\text{A.1})$$

or equivalently

$$X_t^h = h_t S_t. \quad (\text{A.2})$$

The strategy is said to be **self financing** if

$$dX_t^h = \sum_{i=0}^N h_t^i dS_t^i, \quad (\text{A.3})$$

or equivalently

$$X_t^h = h_t dS_t. \quad (\text{A.4})$$

For a given strategy  $h$ , the corresponding **relative portfolio**  $u = (u^1, \dots, u^N)$  is defined by

$$u_t^i = \frac{h_t^i S_t^i}{X_t^h}, \quad i = 0, \dots, N, \quad (\text{A.5})$$

and we will obviously have

$$\sum_{i=0}^N u_t^i = 1.$$

We should, in all honesty, also require some minimal integrability properties for our admissible portfolios, but we will suppress these and some other technical conditions. The reader is referred to the specialist literature for details.

As in the Wiener case, it is often easier to work with the relative portfolio  $u$  than with the portfolio  $h$ . We immediately have the following obvious result.

**Proposition A.1.1** *If  $u$  is the relative portfolio corresponding to a self financing portfolio  $h$ , then we have*

$$dX_t^h = X_t^h \sum_{i=0}^N u_t^i \frac{dS_t^i}{S_t^i}. \quad (\text{A.6})$$

In most market models we have a (locally) risk free asset, and the formal definition is as follows.

**Definition A.1.2** *Suppose that one of the asset price processes, henceforth denoted by  $B$ , has dynamics of the form*

$$dB_t = r_t B_t dt, \quad (\text{A.7})$$

where  $r$  is some adapted random process. In such a case we say that the asset  $B$  is (locally) **risk free**, and we refer to  $B$  as the **bank account**. The process  $r$  is referred to as the corresponding **short rate**.

The term “locally risk free” is more or less obvious. If we are standing at time  $t$  then, since  $r$  is adapted, we know the value of  $r_t$ . We also know  $B_t$ , which implies that already at time  $t$  we know the value  $B_{t+dt} = B_t + r_t B_t dt$  of  $B$  at time  $t + dt$ . The asset  $B$  is thus risk free on the local (infinitesimal) time scale, even if the short rate  $r$  is random. The interpretation is the usual, i.e. we can think of  $B$  as the value of a bank account where we have the short rate  $r$ . Typically we will choose  $B$  as the asset  $S^0$ .

## A.2 Arbitrage

The definition of arbitrage is standard.

**Definition A.2.1** *A portfolio strategy  $h$  is an **arbitrage** strategy on the time interval  $[0, T]$  if the following conditions are satisfied.*

1. *The strategy  $h$  is self financing*
2. *The initial cost of  $h$  is zero, i.e.*

$$X_0^h = 0.$$

3. *At time  $T$  it holds that*

$$\begin{aligned} P(X_T^h \geq 0) &= 1, \\ P(X_T^h > 0) &> 0. \end{aligned}$$

An arbitrage strategy is thus a money making machine which produces positive amounts of money out of nothing. The economic interpretation is that the existence of an arbitrage opportunity signifies a serious case of mispricing in the market, and a minimal requirement of market efficiency is that there are no arbitrage opportunities. The single most important result in mathematical finance is the “first fundamental theorem” which connects absence of arbitrage to the existence of a martingale measure.

**Definition A.2.2** *Consider a market model consisting of  $N+1$  assets  $S^0, \dots, S^N$ , and assume that the **numeraire asset**  $S^0$  has the property that  $S_t^0 > 0$  with probability one for all  $t$ . An equivalent **martingale measure** is a probability measure  $Q$  with the properties that*

1.  *$Q$  is equivalent to  $P$ , i.e.  $Q \sim P$ .*
2. *The normalized price processes  $Z_t^0, \dots, Z_t^N$ , defined by*

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \dots, N,$$

*are (local) martingales under  $Q$ .*

We can now state the main abstract result of arbitrage theory.

**Theorem A.2.1 (The First Fundamental Theorem)** *The market model is free of arbitrage possibilities if and only if there exists a martingale measure  $Q$ .*

**Proof.** This is a very deep result, and the reader is referred to the literature for a proof. ■

**Remark A.2.1** *The First Fundamental Theorem as stated above is a “Folk Theorem” in the sense that it is not stated with all necessary technical conditions. The statement above will however do nicely for our purposes. For a precise statement and an outline of the full proof, see [2]. For the full (extremely difficult) proof see [9].*

We note that if there exists a martingale measure  $Q$ , then it will depend upon the choice of the numeraire asset  $S^0$ , so we should really index  $Q$  as  $Q^0$ . In most cases the numeraire asset will be the bank account  $B$ , and in this case the measure  $Q$ , which more precisely should be denoted by  $Q^B$ , is known as the **risk neutral martingale measure**.

The following useful result is left as an exercise to the reader. It shows that the risk neutral martingale measure  $Q$  is characterized by the fact that, under  $Q$ , the local rate of return of any asset equals the short rate  $r$ .

**Proposition A.2.1** *Assume absence of arbitrage, and let  $Q$  denote a (not necessarily unique) risk neutral martingale measure with  $B$  as numeraire. Let  $S$  denote the arbitrage free price process of an arbitrary asset, underlying or derivative, and assume that the  $S$  dynamics under  $P$  are of the form*

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t,$$

where  $W$  is a  $d$ -dimensional Wiener process,  $\alpha$  is a scalar adapted process, and  $\sigma$  is an adapted row vector process. The following statements are then equivalent.

- The normalized process

$$Z_t = \frac{S_t}{B_t}$$

is a  $Q$  martingale.

- The  $Q$ -dynamics of  $S$  are of the form

$$dS_t = r_t S_t dt + S_t \sigma_t dW_t^Q,$$

where  $W^Q$  is  $Q$ -Wiener.

### A.3 Girsanov and the market price for risk

Let us consider a financial market driven by a  $d$ -dimensional Wiener process  $W$ . We assume the existence of a bank account  $B$  with dynamics

$$dB_t = r_t B_t dt.$$

We also assume absence of arbitrage and we assume that the market uses the (not necessarily unique) risk neutral martingale measure  $Q$  as pricing measure. Let  $S$  denote the arbitrage free price process of an arbitrary asset, underlying or derivative, and assume that the  $S$  dynamics under  $P$  are of the form

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t.$$

or, in more detail,

$$dS_t = \alpha_t S_t dt + S_t \sum_{i=1}^d \sigma_{it} dW_{it}.$$

We denote by  $L$  the likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t.$$

In a purely Wiener driven framework we know that the  $L$ -dynamics are of the form

$$dL_t = L_t \varphi_t^* dW_t,$$

where the **Girsanov kernel**  $\varphi$  is a  $d$ -dimensional column vector process and where  $*$  denotes transpose. From the Girsanov Theorem we know that

$$dW_t = \varphi_t dt + dW_t^Q,$$

where  $W^Q$  is  $Q$ -Wiener, so the  $Q$ -dynamics of  $S$  takes the form

$$dS_t = \{\alpha_t + \sigma_t \varphi_t\} dt + S_t \sigma_t dW_t^Q.$$

From Proposition A.2 we conclude that **the martingale condition**

$$r_t = \alpha_t + \sigma_t \varphi_t.$$

We may now define the so called **market price of risk** vector process  $\lambda$  by

$$\lambda_t = -\varphi_t, \tag{A.8}$$

We thus have the formula

$$\alpha_t - r_t = \lambda_t \sigma_t,$$

or, in more detail,

$$\alpha_t - r_t = \sum_{i=1}^d \lambda_{it} \sigma_{it}.$$

The economic interpretation is that the excess rate of return  $\alpha_t - r_t$  is the sum of the volatilities  $\sigma_{1t}, \dots, \sigma_{dt}$  multiplied by the associated market prices for risk.

## A.4 Martingale Pricing

We now study the possibility of pricing contingent claims. The formal definition of a claim is as follows.

**Definition A.4.1** *Given a stochastic basis  $(\Omega, \mathcal{F}, P, \mathbf{F})$  and a specified point in time  $T$ , often referred to as “the exercise date”) a **contingent  $T$ -claim** is a random variable  $\mathcal{Y} \in \mathcal{F}_T$ .*

The interpretation is that the holder of the claim will obtain the random amount  $\mathcal{Y}$  of money at time  $T$ . We now consider the “primary” or “underlying” market  $S^0, S^1, \dots, S^N$  as given *a priori*, and we fix a  $T$ -claim  $\mathcal{Y}$ . Our task is that of determining a “reasonable” price process  $\Pi_t[\mathcal{Y}]$  for  $\mathcal{Y}$ , and we assume that the primary market is arbitrage free. A main idea is the following.

The derivative should be priced in a way that is **consistent** with the prices of the underlying assets. More precisely we should demand that the extended market  $\Pi[\mathcal{Y}], S^0, S^1, \dots, S^N$  is free of arbitrage possibilities.

In this approach we thus demand that there should exist a martingale measure  $Q$  for the extended market  $\Pi[\mathcal{Y}], S^0, S^1, \dots, S^N$ . Letting  $Q$  denote such a measure, assuming enough integrability, and applying the definition of a martingale measure we obtain

$$\frac{\Pi_t[\mathcal{Y}]}{S_t^0} = E^Q \left[ \frac{\Pi_T[\mathcal{Y}]}{S_T^0} \middle| \mathcal{F}_t \right] = E^Q \left[ \frac{\mathcal{Y}}{S_T^0} \middle| \mathcal{F}_t \right] \quad (\text{A.9})$$

where we have used the fact that, in order to avoid arbitrage at time  $T$  we must have  $\Pi_T[X] = X$ . We thus have the following result.

**Theorem A.4.1 (General Pricing Formula)** *The arbitrage free price process for the  $T$ -claim  $\mathcal{Y}$  is given by*

$$\Pi_t[\mathcal{Y}] = S_t^0 E^Q \left[ \frac{\mathcal{Y}}{S_T^0} \middle| \mathcal{F}_t \right], \quad (\text{A.10})$$

where  $Q$  is the (not necessarily unique) martingale measure for the *a priori* given market  $S^0, S^1, \dots, S^N$ , with  $S^0$  as the numeraire.

Note that different choices of  $Q$  will generically give rise to different price processes. In particular we note that if we assume that if  $S^0$  is the money account

$$S_t^0 = S_0^0 \cdot e^{\int_0^t r(s) ds},$$

where  $r$  is the short rate, then (A.10) reduced to the familiar “risk neutral valuation formula”.

**Theorem A.4.2 (Risk Neutral Valuation Formula)**

*Assuming the existence of a short rate, the pricing formula takes the form*

$$\Pi_t[\mathcal{Y}] = E^Q \left[ e^{-\int_t^T r(s) ds} \mathcal{Y} \middle| \mathcal{F}_t \right]. \quad (\text{A.11})$$

where  $Q$  is a (not necessarily unique) martingale measure with the money account as the numeraire.

The pricing formulas (A.10) and (A.11) are very nice, but it is clear that if there exists more than one martingale measure (for the chosen numeraire), then the formulas do not provide a unique arbitrage free price for a given claim  $\mathcal{Y}$ . It is thus natural to ask under what conditions the martingale measure is unique, and this turns out to be closely linked to the possibility of hedging contingent claims.

## A.5 Hedging

Consider a market model  $S^0, \dots, S^N$  and a contingent  $T$ -claim  $\mathcal{Y}$ .

**Definition A.5.1** *If there exists a self financing portfolio  $h$  such that the corresponding value process  $X^h$  satisfies the condition*

$$X_T^h = \mathcal{Y}, \quad P - a.s. \quad (\text{A.12})$$

*then we say that  $h$  replicates  $\mathcal{Y}$ , that  $h$  is a hedge against  $\mathcal{Y}$ , or that  $\mathcal{Y}$  is attained by  $h$ . If, for every  $T$ , all  $T$ -claims can be replicated, then we say that the market is complete.*

Given the hedging concept, we now have a second approach to pricing. Let us assume that  $\mathcal{Y}$  can be replicated by  $h$ . Since the holding of the derivative contract and the holding of the replicating portfolio are equivalent from a financial point of view, we see that price of the derivative must be given by the formula

$$\Pi_t[\mathcal{Y}] = X_t^h, \quad (\text{A.13})$$

since otherwise there would be an arbitrage possibility (why?).

We now have two obvious problems.

- What will happen in a case when  $\mathcal{Y}$  can be replicated by two different portfolios  $g$  and  $h$ ?
- How is the formula (A.13) connected to the previous pricing formula (A.10)?

To answer these question, let us assume that the market is free of arbitrage, and let us also assume at the  $T$  claim  $X$  is replicated by the portfolios  $g$  and  $h$ . We choose the bank account  $B$  as the numeraire and consider a fixed martingale measure  $Q$ . Since  $Q$  is a martingale measure for the underlying market  $S^0, \dots, S^N$ , it is easy to see that this implies that  $Q$  is also a martingale measure for  $X^g$  and  $X^h$  in the sense that  $X^h/B$  and  $X^g/B$  are  $Q$  martingales. Using this we obtain

$$\frac{X_t^h}{B_t} = E^Q \left[ \frac{X_T^h}{B_T} \middle| \mathcal{F}_T \right]$$

and similarly for  $X^g$ . Since, by assumption, we have  $X_T^h = \mathcal{Y}$  we thus have

$$X_t^h = E^Q \left[ \mathcal{Y} \frac{B_t}{B_T} \middle| \mathcal{F}_T \right],$$

which will hold for any replicating portfolio and for any martingale measure  $Q$ . Assuming absence of arbitrage we have thus proved the following.

- If  $\mathcal{Y}$  is replicated by  $g$  and  $h$ , then

$$X_t^h = X_t^g, \quad t \geq 0.$$

- For an attainable claim, the value of the replicating portfolio coincides with the risk neutral valuation formula, i.e.

$$X_t^h = E^Q \left[ e^{-\int_t^T r_s ds} \mathcal{Y} \middle| \mathcal{F}_T \right].$$

From (A.11) it is obvious that every claim  $\mathcal{Y}$  will have a unique price if and only if the martingale measure  $Q$  is unique. On the other hand, it follows from the alternative pricing formula (A.13) that there will exist a unique price for every claim if every claim can be replicated. The following result is therefore not surprising.

**Theorem A.5.1 (Second Fundamental Theorem)** *Given a fixed numeraire  $S^0$ , the corresponding martingale measure  $Q^0$  is unique if and only if the market is complete.*

**Proof.** We have already seen above that if the market is complete, then the martingale measure is unique. The other implication is a very deep result, and the reader is referred to [9]. ■

## A.6 Stochastic Discount Factors

In the previous sections we have seen that we can price a contingent  $T$ -claim  $\mathcal{Y}$  by using the formula

$$\Pi_0[\mathcal{Y}] = E^Q \left[ e^{-\int_t^T r_s ds} \mathcal{Y} \right], \quad (\text{A.14})$$

where  $Q$  is a martingale measure with the bank account as a numeraire. In many applications of the theory, in particular in equilibrium theory, it is very useful to write this expected value directly under the objective probability measure  $P$  instead of under  $Q$ . This can easily be obtained by using the likelihood process  $L$ , where a usual  $L$  is defined on the interval  $[0, T]$  through

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t. \quad (\text{A.15})$$

We can then write (A.14) as

$$\Pi_0[\mathcal{Y}] = E^P [B_T^{-1} L_T \mathcal{Y}],$$

which naturally leads us to the following definition.

**Definition A.6.1** *Assume the existence of a short rate  $r$ . For any fixed martingale measure  $Q$ , let the likelihood process  $L$  be defined by (A.15). The **stochastic discount factor** (SDF) process  $M$ , corresponding to  $Q$ , is defined as*

$$M_t = B_t^{-1} L_t. \quad (\text{A.16})$$



We thus see that there is a one-to-one correspondence between martingale measures and stochastic discount factors, and we can write the pricing formula as

$$\Pi_0[\mathcal{Y}] = E^P[M_T\mathcal{Y}]$$

This gives us the arbitrage free price at  $t = 0$  as a  $P$ -expectation, but the result can easily be generalized to arbitrary times  $t$ .

**Proposition A.6.1** *Assume absence of arbitrage. With notation as above, the following hold.*

- For any sufficiently integrable  $T$ -claim  $\mathcal{Y}$ , the arbitrage free price is given by

$$\Pi_t[\mathcal{Y}] = \frac{1}{M_t} E^P[M_T\mathcal{Y} | \mathcal{F}_t]. \quad (\text{A.17})$$

- For any arbitrage free asset price process  $S$  (derivative or underlying) the process

$$M_t S_t \quad (\text{A.18})$$

is a (local)  $P$ -martingale.

- The  $P$ -dynamics of  $M$  are given by

$$dM_t = -r_t M_t dt + \frac{1}{B_t} dL_t. \quad (\text{A.19})$$

**Proof.** Use the Abstract Bayes' formula to prove the first two statements. The remaining details of the proof are left to the reader. ■

The following easy corollary is used a lot in equilibrium theory.

**Proposition A.6.2** *Assume that the likelihood process  $L = dQ/dP$  has dynamics*

$$dL_t = L_t \varphi_t dW_t,$$

Then the  $M$  dynamics can be written as

$$dM_t = -r_t M_t dt + M_t \varphi_t dW_t. \quad (\text{A.20})$$

**Proof.** Exercise for the reader. ■

Using (A.8) we can also write the  $M$  dynamics as

$$dM_t = -r_t M_t dt - M_t \lambda_t dW_t. \quad (\text{A.21})$$

where  $\lambda$  is the market price of risk vector process.

**Remark A.6.1** *The point of Proposition A.6.2 is that the short rate  $r$  as well as the Girsanov kernel can be identified from the dynamics of the stochastic discount factor.*

Although SDFs and martingale measures are logically equivalent, it is often convenient to be able to switch from one to the other. In asset pricing and equilibrium theory it is often natural to use the SDF formalism, whereas it seems more convenient to use the language of martingale measures in connection with arbitrage theory and pricing of contingent claims.

## A.7 Dividends

So far we have assumed that all assets are non-dividend paying- We now extend the theory and to that end we consider a market consisting of the usual bank account  $B$  as well as  $N$  risky assets. The novelty is that all assets are allowed to pay dividends so we need to formalize this idea. As usual we denote by  $S_t^i$  the price at time  $t$  of asset number  $i$ , and we denote by  $D^i$  the **cumulative dividend process** of asset number  $i$ . The interpretation is that  $D_t^i$  is the total amount of dividends that have been paid by holding one unit of the asset over the time period  $[0, t]$ . Intuitively this means that over an infinitesimal interval  $[t, t + dt]$  the holder of the asset will obtain the amount  $dD_t^i = D_{t+dt}^i - D_t^i$ . For simplicity we also assume that the trajectory of  $D^i$  is continuous.

A market of this kind thus consist of the bank account  $S^0 = B$  and a collection of **price dividend pairs**  $(S^1, D^1), \dots, (S^N, D^N)$ . We now have the following extension of the previous theory. For details, see [2].

**Definition A.7.1** *A portfolio strategy is an  $N+1$  dimensional adapted (row vector) process  $h = (h^0, h^1, \dots, h^N)$ . For a given strategy  $h$ , the corresponding value process  $X^h$  is defined by*

$$X_t^h = \sum_{i=0}^N h_t^i S_t^i, \quad (\text{A.22})$$

*The strategy is said to be self financing if*

$$dX_t^h = \sum_{i=0}^N h_t^i dG_t^i, \quad (\text{A.23})$$

*where the gain process  $G$  is defined by*

$$dG_t^i = dS_t^i + dD_t^i,$$

*For a given strategy  $h$ , the corresponding relative portfolio  $u = (u^0, \dots, u^N)$  is defined by*

$$u_t^i = \frac{h_t^i S_t^i}{X_t^h}, \quad i = 0, \dots, N, \quad (\text{A.24})$$

The interpretation of the self financing condition (A.23) is clear. If you hold  $h$  units of an asset over the interval  $[t, t + dt]$  then you gain (or loose)  $h_t dS_t$  because of the price change and you get  $h_t dD_t$  in dividends.

**Proposition A.7.1** *If  $u$  is the relative portfolio corresponding to a self financing portfolio  $h$ , then we have*

$$dX_t^h = X_t^h \sum_{i=0}^N u_t^i \frac{dS_t^i + dD_t^i}{S_t^i}. \quad (\text{A.25})$$

The definition of arbitrage is exactly as in the non-dividend case, but for the concept of a martingale measure we need a small variation.

**Definition A.7.2** *An equivalent martingale measure is a probability measure  $Q$  with the properties that*

1.  $Q$  is equivalent to  $P$ , i.e.  $Q \sim P$ .
2. The **normalized gain processes**  $G_t^{Z0}, \dots, G_t^{ZN}$ , defined by

$$G_t^{Zi} = \frac{S_t^i}{B_t} + \int_0^t \frac{1}{B_s} dD_s^i, \quad i = 0, \dots, N,$$

are (local) martingales under  $Q$ .

The martingale property of  $\widehat{G}$  has a very natural interpretation. For a price-dividend pair  $(S, D)$  we obtain, after some reshuffling of terms.

$$S_t = E^Q \left[ \int_t^T e^{-\int_t^s r_u du} dD_s + e^{-\int_t^T r_s ds} S_T \middle| \mathcal{F}_t \right].$$

This is a risk neutral valuation formula, which says that the stock price at time  $t$  equals the arbitrage free value at  $t$  of the stock price  $S_T$  plus the sum (or rather integral) of the arbitrage free value at  $t$  of all dividends over the interval  $[t, T]$ .

In the present setting the first fundamental theorem takes the following form.

**Theorem A.7.1 (The First Fundamental Theorem)** *The market model is free of arbitrage possibilities if and only if there exists a martingale measure  $Q$ .*

The pricing formulas are as expected.

**Proposition A.7.2** *The arbitrage free price of a  $T$ -claim  $\mathcal{Y}$  is given by*

$$\Pi_t[X] = E^Q \left[ e^{-\int_t^T r_s ds} \mathcal{Y} \middle| \mathcal{F}_t \right].$$

## A.8 Consumption

We now introduce consumption into the theory. For simplicity we will only consider consumption programs which can be represented in terms of a **consumption rate process**  $c$ . The interpretation is that over an infinitesimal interval  $[t, t + dt]$  the economic agent consumes the amount  $c_t dt$ , so the dimension of  $c_t$  is “consumption per unit time”. We define portfolios  $h^0, \dots, h^N$ , portfolio value  $X$ , and martingale measure, as in the previous section.

Given a market of the form  $B, (S^1, D^1), \dots, (S^N, D^N)$  we say that a portfolio  $h$  is self financing for the consumption process  $c$  if

$$dX_t^h = \sum_{i=0}^N h_t^i \{dS_t^i + dD_t^i\} - c_t dt.$$

or, in terms of relative weights

$$dX_t^h = X_t^h \sum_{i=0}^N u_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t dt.$$

The pricing theory for contingent claims can easily be extended to include consumption.

**Proposition A.8.1** *The arbitrage free price, at time  $t$ , for the consumption process  $c$ , restricted to the interval  $[t, T]$  is given by*

$$\Pi_t [c] = E^Q \left[ \int_t^T e^{-\int_t^s r_s ds} c_s ds \middle| \mathcal{F}_t \right].$$

Using the stochastic discount factor  $M = B^{-1}L$  we can write this as

$$\Pi_t [c] = \frac{1}{M_t} E^P \left[ \int_t^T M_s c_s ds \middle| \mathcal{F}_t \right].$$

Furthermore, if the market is complete, then every consumption stream can be replicated by a self financing portfolio.

## A.9 Replicating a consumption process

Consider a market  $B, (S^1, D^1), \dots, (S^N, D^N)$  and a consumption process  $c$ . If the market is complete, then it follows from Proposition A.8.1 that  $c$  can be replicated with a self financing portfolio. This result, however, is an abstract existence result, but in a concrete case it will not tell us what the replicating portfolio looks like. We will now present a proposition which can be used to construct the replicating portfolio.

We start with a small lemma which show that, as could be expected, the property of being self financing is invariant under normalization.

**Lemma A.9.1 (Invariance Lemma)** Consider a market as above, a consumption process  $c$ , and a portfolio process  $h = (h^0, h^1, \dots, h^N)$ . Let  $X$  denote the corresponding value process. Then it holds that  $h$  is self financing for  $c$  if and only if

$$dX_t^Z = \sum_{i=1}^N h_t^i dG_t^{Z^i} - c_t^z dt$$

where the normalized value process  $X^Z$  and the normalized consumption process  $c^Z$  are defined by

$$\begin{aligned} X_t^Z &= \frac{X_t}{B_t}, \\ c_t^z &= \frac{c_t}{B_t}. \end{aligned}$$

**Proof.** The proof is left to the reader as an exercise. ■

**Proposition A.9.1** Consider the market  $B, (S^1, D^1), \dots, (S^N, D^N)$  and a consumption process  $c$ . Define the process  $K$  by

$$K_t = E^Q \left[ \int_0^T c_s^z ds \middle| \mathcal{F}_t \right],$$

with  $c^z$  as in the Lemma above. Assume that there exist processes  $h^1, \dots, h^N$  such that

$$dK_t = \sum_{i=1}^N h_t^i dG_t^{Z^i}$$

Then  $c$  can be replicated by the portfolio  $h^0, h^1, \dots, h^N$ , with  $h^1, \dots, h^N$ , as above and  $h^0$  defined by

$$h_t^0 = K_t - \sum_{i=1}^N h_t^i Z_t^i - \int_0^t c_s^z ds.$$

**Proof.** The normalized portfolio value of  $h^0, h^1, \dots, h^N$  is, by definition, given by

$$X_t^Z = h_0 \cdot 1 + \sum_{i=1}^N h_t^i Z_t^i$$

so, using the definition of  $h^0$ , we have

$$X_t^Z = K_t - \int_0^t c_s^z ds.$$

From the assumption  $dK_t = \sum_i h_t^i dG_t^{zi}$  we thus obtain

$$dX_t^Z = \sum_i h_t^i dG_t^{zi} - c_t^z dt.$$

This shows that  $h$  is self financing in the normalized economy, and by Lemma A.9.1, it is also self financing in nominal terms. ■

## A.10 Exercises

**Exercise A.1** *Prove Lemma A.9.1.*

## A.11 Notes

The martingale approach to arbitrage pricing was developed in [18], [26], and [19]. It was then extended by, among others, [13], [8], [32], and [9]. Stochastic discount factors are treated in most modern textbooks on asset pricing such as [3] and [12].

# Appendix B

## The Conditional Density

In this chapter we will present some further results from filtering theory, such as the stochastic PDE for the conditional density and the Zakai equation for the un-normalized density. These results are important but also more technical than the previous results, so this appendix can be skipped in a first reading.

### B.1 The evolution of the conditional density

The object of this section is to derive a stochastic PDE describing the evolution of the conditional density of the state process  $Y$ . This theory is, at some points, quite technical, so we only give the basic arguments. For details, see [1]. We specialize to the Markovian setting

$$dY_t = a(Y_t)dt + c(Y_t)dW_t^0, \quad (\text{B.1})$$

$$dZ_t = b(Y_t)dt + dW_t^1 \quad (\text{B.2})$$

where  $W^0$  and  $W^1$  are independent Wiener processes. We recall from (12.16)-(12.17) that for any real valued function  $f : R \rightarrow R$ , the filter equations for  $\pi_t(f) = E[f(Y_t) | \mathcal{F}_t^Z]$  are then given by

$$d\pi_t(f) = \pi_t(\mathcal{A}f) dt + \{\pi_t(fb) - \pi_t(f) \cdot \pi_t(b)\} d\nu_t, \quad (\text{B.3})$$

$$d\nu_t = dZ_t - \pi_t(b) dt. \quad (\text{B.4})$$

where the infinitesimal generator  $\mathcal{A}$  is defined by

$$\mathcal{A} = a(y) \frac{\partial}{\partial y} + \frac{1}{2} c^2(y) \frac{\partial^2}{\partial y^2}.$$

We now view  $f$  as a ‘test function’ varying within a large class  $\mathcal{C}$  of test functions. The filter equation (B.3) will then, as  $f$  varies over  $\mathcal{C}$  determine the entire conditional distribution  $\mathcal{L}(Y_t | \mathcal{F}_t^Z)$ .

Let us now **assume that  $Y$  has a conditional density process  $p_t(y)$** , w.r.t. Lebesgue measure, so that

$$\pi_t(f) = E [f(Y_t) | \mathcal{F}_t^Z] = \int_R f(y) p_t(y) dy$$

In order to have a more suggestive notation we introduce a natural pairing (“inner product”) denoted by  $\langle \cdot, \cdot \rangle$  for any smooth real valued functions  $g$  and  $f$  (where  $f$  has compact support). This is defined by

$$\langle g, f \rangle = \int_R f(y) g(y) dy.$$

We can thus write

$$\pi_t(f) = \langle p_t, f \rangle,$$

and with this notation, the filter equation takes the form

$$d\langle p_t, f \rangle = \langle p_t, \mathcal{A}f \rangle dt + \{ \langle p_t, fb \rangle - \langle p_t, f \rangle \langle p_t, b \rangle \} d\nu_t$$

We can now dualize this (see the exercises) to obtain

$$d\langle p_t, f \rangle = \langle \mathcal{A}^* p_t, f \rangle dt + \{ \langle bp_t, f \rangle - \langle p_t, f \rangle \langle p_t, b \rangle \} d\nu_t$$

where  $\mathcal{A}^*$  is the adjoint operator:

$$\mathcal{A}^* p(y) = -\frac{\partial}{\partial y} [a(y)p(y)] + \frac{1}{2} \frac{\partial^2}{\partial y^2} [c^2(y)p(y)]$$

If this holds for all test functions  $f$  we are led to the following result. See [1] for the full story.

**Theorem B.1.1 (Kushner-Stratonovich)** *Assume that  $Y$  has a conditional density process  $p_t(y)$  w.r.t. Lebesgue measure. Under suitable technical conditions, the density will satisfy the following stochastic partial differential equation (SPDE)*

$$dp_t(y) = \mathcal{A}^* p_t(y) dt + p_t(y) \left\{ b(y) - \int_R b(y) p_t(y) dy \right\} d\nu(t), \quad (\text{B.5})$$

$$d\nu_t = dZ_t - \left( \int_R b(y) p_t(y) dy \right) dt. \quad (\text{B.6})$$

As we noted above, this is an SPDE for the conditional density. In order to connect to more familiar topics, we remark that if the observation dynamics have the form

$$dZ_t = b(Y_t) dt + \sigma dW_t,$$

then the SPDE would have the form

$$\begin{aligned} dp_t(y) &= \mathcal{A}^* p_t(y) dt + \frac{p_t(y)}{\sigma} \left\{ b(y) - \int_R b(y) p_t(y) dy \right\} d\nu(t), \\ d\nu_t &= \frac{1}{\sigma} \left\{ dZ_t - \left( \int_R b(y) p_t(y) dy \right) dt \right\}. \end{aligned}$$



If we now let  $\sigma \rightarrow +\infty$ , which intuitively means that in the limit we only observe noise, then the filter equation degenerates to

$$\frac{\partial}{\partial t} p_t(y) = \mathcal{A}^* p_t(y),$$

which is the Fokker-Planck equation for the unconditional density.

## B.2 Estimation of a likelihood process

The main result in this Section is Proposition B.2.2 and Corollary B.9, which will be important when we on derive the Zakai equation in the next section, and the reader who wishes to proceed directly to the Zakai equation can skip the present section and simply accept Proposition B.2.2 and the Corollary dogmatically. The Theorem is, however, of considerable importance also in statistics, so for the reader with more general interests, we provide the full story.

Consider a measurable space  $\{\Omega, \mathcal{F}\}$  and two probability measures,  $P_0$  and  $P_1$  on this space. Let us now consider a case of hypothesis testing in this framework. We have two hypotheses,  $H_0$  and  $H_1$  with the interpretation

$$\begin{aligned} H_0 : & \quad P_0 \text{ holds,} \\ H_1 : & \quad P_1 \text{ holds.} \end{aligned}$$

If  $P_1 \ll P_0$ , then we know from Neyman-Pearson that the appropriate test variable is given by the likelihood

$$L = \frac{dP_1}{dP_0}, \quad \text{on } \mathcal{F}$$

and that the optimal test is of the form

$$\text{Reject } H_0 \text{ if } L \geq R,$$

where  $R$  is a suitably chosen constant.

The arguments above hold as long as we really have access to all the information contained in  $\mathcal{F}$ . If, instead of  $\mathcal{F}$ , we only have access to the information in a smaller sigma algebra  $\mathcal{G} \subset \mathcal{F}$ , then we cannot perform the test above, since  $L$  is not determined by the information in  $\mathcal{G}$  (i.e.  $L$  is not  $\mathcal{G}$ -measurable).

From an abstract point of view, this problem is easily solved by simply applying Neyman-Pearson to the space  $\{\Omega, \mathcal{G}\}$  instead of  $\{\Omega, \mathcal{F}\}$ . It is then obvious that the optimal test variable is given by

$$\widehat{L} = \frac{dP_1}{dP_0}, \quad \text{on } \mathcal{G},$$

and the question is now how  $\widehat{L}$  is related to  $L$ .

This question is answered by the following standard result.

**Proposition B.2.1** *With notation as above, we have*

$$\widehat{L} = E^0 [L | \mathcal{G}], \quad (\text{B.7})$$

where  $E^0$  denotes expectation under  $P_0$ .

We will now study the structure of  $L$ , when  $L$  is generated by a Girsanov transformation and we also have a filtered space, so information increases over time. To this end we consider a filtered probability space  $\{\Omega, \mathcal{F}, P_0, \mathbf{F}\}$  and two  $\mathbf{F}$  adapted processes  $Z$  and  $h$ . We assume the following

- $Z$  is an  $\mathbf{F}$ -Wiener process under  $P_0$ .
- The process  $h$  is  $\mathbf{F}$ -adapted and  $P_0$ -independent of  $Z$ .

We now introduce some useful notation connected to Girsanov transformations.

**Definition B.2.1** *For any  $\mathbf{F}$ -adapted process  $g$  we define the process  $L(g)$  by*

$$L_t(g) = e^{\int_0^t g_s dZ_s - \frac{1}{2} \int_0^t g_s^2 ds}. \quad (\text{B.8})$$

We see that  $L(g)$  is simply the likelihood process for a Girsanov transformation if we use  $g$  as the the Girsanov kernel.

Assuming that  $L(h)$  is a true martingale we may now define the measure  $P_1$  by setting

$$L_t(h) = \frac{dP_1}{dP_0}, \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T$$

where  $T$  is some fixed time horizon.

From the Girsanov Theorem we know that we can write

$$dZ_t = h_t dt + dW_t^1,$$

where  $W^1$  is an  $P_1$ -Wiener process. We thus see that we have the following situation.

- Under  $P_0$ , the process  $Z$  is a Wiener process without drift.
- Under  $P_1$ , the process  $Z$  has the drift  $h$ .

If we now interpret the process  $h$  as a signal, we have the following tho hypotheses:

$H_0$  : We have no signal. We only observe Wiener noise.

$H_1$  : We observe the signal  $h$ , disturbed by the Wiener noise.

The task at hand is to test  $H_0$  against  $H_1$  sequentially over time, assuming that we can only observe the process  $Z$ . From Proposition B.2.1 we know that, at time  $t$ , the optimal test statistics is given by  $\widehat{L}_t(h)$ , defined by

$$\widehat{L}_t(h) = E^0 [L_t(h) | \mathcal{F}_t^Z],$$

and the question is how we compute this entity. The answer is given by the following beautiful result.

**Proposition B.2.2** *With notation as above we have*

$$\widehat{L}_t(h) = L_t(\widehat{h}),$$

where

$$\widehat{h}_t = E^1 [h_t | \mathcal{F}_t^Z].$$

Note that  $\widehat{L}_t(h)$  is a  $P_0$ -expectation, whereas  $\widehat{h}_t$  is a  $P_1$ -expectation. Before going on to the proof, we remark that this is a separation result. It says that the filtered “detector”  $\widehat{L}$  can be separated into two parts: The (unfiltered) detector  $L$ , and the  $P_1$ -filter estimate  $\widehat{h}$ .

**Proof.** From the definition of  $L_t(h)$  we have

$$dL_t(h) = L_t(h)h_t dZ_t,$$

and we observe the  $P_0$  Wiener process  $Z$ . From Theorem 12.4.1 we then have

$$d\widehat{L}_t(h) = \pi_t^0(L(h)h) dZ_t,$$

where upper case index denotes expectation under  $P_0$ . We can write this as

$$d\widehat{L}_t(h) = \widehat{L}_t(h)\eta_t dZ_t,$$

where

$$\eta_t = \frac{\pi_t^0(L(h)h)}{\pi_t^0(L(h))},$$

so we have in fact

$$\widehat{L}_t(h) = L_t(\eta).$$

On the other hand we have

$$\eta_t = \frac{E^0 [L_t(h)h_t | \mathcal{F}_t^Z]}{E^0 [L_t(h) | \mathcal{F}_t^Z]}$$

so from the abstract Bayes formula we obtain

$$\eta_t = E^1 [h_t | \mathcal{F}_t^Z] = \pi_t^1(h). \quad \blacksquare$$

For future use we note the following Corollary which in fact is a part of the proof above.

**Corollary B.2.1** *We have*

$$d\widehat{L}_t(h) = \pi_t^1(h) \widehat{L}_t(h) dZ_t. \tag{B.9}$$

### B.3 Un-normalized filter estimates

We consider again a filtered probability space  $\{\Omega, \mathcal{F}, P, \mathbf{F}\}$  and a Markovian filter model of the form

$$dY_t = a(Y_t)dt + c(Y_t)dW_t^0, \quad (\text{B.10})$$

$$dZ_t = b(Y_t)dt + dW_t \quad (\text{B.11})$$

where  $W^0$  and  $W$  are independent. In Section B.1 we studied the same model, and we derived the Kushner-Stratonovich equation (B.5) for the density  $p_t(y)$ . We now present an alternative approach to the filtering problem along the following lines.

- Perform a Girsanov transformation from  $P$  to a new measure  $P_0$  such that  $Y$  and  $Z$  are independent under  $P_0$ .
- Compute filter estimates under  $P_0$ . This should be easy, due to the independence.
- Transform the results back to  $P$  using the abstract Bayes formula.
- This will lead to a study of the so-called **un-normalized** density  $q_t(y)$ .
- We will derive an SPDE, known as the Zakai equation, for the un-normalized density  $q$ . This equation is, in many ways, much nicer than the Kushner-Stratonovich equation for the density  $p_t(y)$ .

#### B.3.1 The basic construction

Consider a probability space  $\{\Omega, \mathcal{F}, P_0\}$  as well as two independent  $P_0$ -Wiener processes  $Z$  and  $W^0$ . We define filtration  $\mathbf{F}$  by

$$\mathcal{F}_t = \mathcal{F}_t^Z \vee \mathcal{F}_\infty^{W^0},$$

and we define the process  $Y$  by

$$dY_t = a(Y_t)dt + c(Y_t)dW_t^0.$$

We now define the likelihood process  $L$  by

$$\begin{aligned} dL_t &= L_t b(Y_t) dt dZ_t, \\ L_0 &= 1, \end{aligned}$$

and we define  $P$  by

$$L_t = \frac{dP}{dP_0}, \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T$$

for some fixed horizon  $T$ . From the Girsanov Theorem it now follows that the we can write

$$dZ_t = b(Y_t)dt + dW_t$$

where  $W$  is a  $(P, \mathbf{F})$ -Wiener process. In particular, the process  $W$  is independent of  $\mathcal{F}_0 = \mathcal{F}_\infty^{W^0}$ , so  $W$  and  $W^0$  are independent under  $P$ . Since  $L_0 = 1$  we have  $P = P_0$  on  $\mathcal{F}_0$ , and since  $\mathcal{F}_0 = \mathcal{F}_\infty^{W^0}$  we see that  $(W^0, Y)$  has the same distribution under  $P$  as under  $P_0$ .

The end result of all this is that under  $P$  we have our standard model

$$\begin{aligned} dY_t &= a(Y_t)dt + c(Y_t)dW_t^0, \\ dZ_t &= b(Y_t)dt + dW_t. \end{aligned}$$

### B.3.2 The Zakai equation

We define  $\pi_t(f)$  as usual by

$$\pi_t(f) = E^P [f(Y_t) | \mathcal{F}_t^Z],$$

and the abstract Bayes formula we then have

$$\pi_t(f) = \frac{E^0 [L_t f(Y_t) | \mathcal{F}_t^Z]}{E^0 [L_t | \mathcal{F}_t^Z]}.$$

**Definition B.3.1** *The un-normalized estimate  $\sigma_t(f)$  is defined by*

$$\sigma_t(f) = E^0 [L_t f(Y_t) | \mathcal{F}_t^Z].$$

We have thus derived the Kallianpur-Striebel formula.

**Proposition B.3.1 (Kallianpur-Striebel)** *The standard filter estimate  $\pi_t(f)$  and the un-normalized estimate  $\sigma_t(f)$  are related by the formula*

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}. \tag{B.12}$$

It turns out that  $\sigma_t(f)$  is easier to study than  $\pi_t(f)$ , so we now go on to derive the dynamics of  $\sigma_t(f)$ .

This is in fact quite easy. From the Kallianpur-Striebel formula we have

$$\sigma_t(f) = \pi_t(f) \cdot \sigma_t(1).$$

From the FKK Theorem we have

$$\begin{aligned} d\pi_t(f) &= \pi_t(\mathcal{A}f) dt + \{\pi_t(fb) - \pi_t(f) \cdot \pi_t(b)\} d\nu_t, \\ d\nu_t &= dZ_t - \pi_t(b) dt, \end{aligned}$$

and from Corollary B.2.1 have

$$d\sigma_t(1) = \sigma_t(1) \pi_t(b) dZ_t.$$

We can now apply the Ito formula to the product  $\pi_t(f) \cdot \sigma_t(1)$ . This leads to calculations which, at first sight, look rather forbidding. It turns out, however, that there are a surprising large number of cancellations in these calculations and in the end we obtain the following result.

**Theorem B.3.1 (The Zakai Equation)** *The un-normalized estimate  $\sigma_t(f)$  satisfies the Zakai equation*

$$d\sigma_t(f) = \sigma_t(\mathcal{A}f) dt + \sigma_t(b) dZ_t. \quad (\text{B.13})$$

We note that the Zakai equation has a much simpler structure than the corresponding FKK equation. Firstly, the Zakai equation is driven directly by the observation process  $Z$ , rather than by the innovation process  $\nu$ . Secondly, the non-linear (product) term  $\pi_t(f) \cdot \pi_t(b) d\nu_t$  is replaced by the term  $\sigma_t(b) dZ_t$  which does not involve  $f$ .

### B.3.3 The SPDE for the un-normalized density

Let us now assume that there exists an unnormalized density process  $q_t(y)$ , with interpretation

$$\sigma_t(f) = \int_R f(y) q_t(y) dy.$$

We can then dualize the Zakai equation, exactly like we did in the derivation of the Kushner-Stratonovich equation, to obtain the following result.

**Proposition B.3.2** *Assuming that there exists an un-normalized density  $q_t(y)$ , we have the following SPDE.*

$$dq_t(y) = \mathcal{A}^* q_t(y) dt + b(y) q_t(y) dZ_t. \quad (\text{B.14})$$

As dot the Zakai equation, we note that the SPDE above for the un-normalized density is much simpler than the Kushner-Stratonovich equation for the normalized density  $p_t(y)$ .

## B.4 Exercises

**Exercise B.1** *Consider a probability space  $(\Omega, \mathcal{F}, P)$  and the filtering model*

$$dZ_t = X dt + dW_t$$

where  $X$  is a random variable with distribution function  $F$ , and  $W$  is a Wiener process which is independent of  $X$ . As usual we observe  $Z$ . In Exercise 12.2 we saw that a naive application of the FKK equation produced an infinite dimensional filter. The object of this exercise is to show that we can do better.

Define, therefore, the functions  $f : R \times R_+ \rightarrow R$  and  $g : R \times R_+ \rightarrow R$  by

$$\begin{aligned} f(t, z) &= \int_R x e^{-\frac{t}{2} \left(x - \frac{z}{t}\right)^2} dF(x), \\ g(t, z) &= \int_R e^{-\frac{t}{2} \left(x - \frac{z}{t}\right)^2} dF(x), \end{aligned}$$

and show that

$$E^P [X | \mathcal{F}_t^Z] = \frac{f(t, Z_t)}{g(t, Z_t)}$$

**Hint:** Perform a Girsanov transformation from  $P$  to a new measure  $Q$  so that  $Z$  and  $X$  are  $Q$ -independent. You may then use (a generalization of) the fact that if  $\xi$  and  $\eta$  are independent random variables where  $\xi$  has distribution function  $F$  then, for any function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ , we have

$$E [H(\xi, \eta) | \sigma \{\eta\}] = h(\eta),$$

where  $h$  is defined by

$$h(y) = \int_{\mathbb{R}} H(x, y) dF(x)$$

## B.5 Notes

A very complete account of FKK theory as well as un-normalized filtering is given in the (advanced) book [1].





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